
Contents

Prologue	15
-----------------------	----

1

Introduction. Classical Approximation	19
--	----

1.1 Introduction	19
1.2 Posing the Problem	20
1.3 The Direct Stiffness (or Displacement) Method	23
1.4 Matrix Formulation	24
1.5 Conclusions	29
1.6 Application examples	29

2

Foundations: Differential and Integral Formulations. Approximation. Elements.	45
--	----

2.1 Introduction	45
2.2 The State or Field Equation. Differential or Strong Formulation	46
2.3 Direct Formulation	48
2.4 The Principle of Virtual Work	49
2.5 Energy Formulation	50
2.6 Equivalence of Integral and Differential Formulations	51
2.7 Approximation	52
2.8 Galerkin's Method	58
2.9 Spline Functions	61
2.10 The Concept of an Element. Shape Functions	68
2.11 Structure Stiffness Matrix and Load Vector. Calculation of Displacements ...	75

2.12	Timoshenko's Beam	84
------	-------------------------	----

3

The Direct Stiffness Method.....		93
---	--	-----------

3.1	Introduction	93
3.2	Coordinate Systems	93
3.3	Transformation of Coordinates. Axes Rotation	97
3.4	Assembling the System of Equations	101
3.5	The Boundary Conditions	104
3.6	Calculation of Displacements	111
3.7	Calculating Reactions and Internal Forces	128
3.8	Examples	129

4

General Approach for the FEM.....		165
--	--	------------

4.1	Introduction	165
4.2	Differential Approach to the Boundary Value Problem	165
4.3	Integral Approach to the Boundary Value Problem	169
4.4	Approximation	172
4.5	The Finite Element Method	175
4.6	Global Features. Application of Essential Boundary Conditions	182
4.7	Application of FEM in Solving Linear Elastic Problems	183
4.8	Steady-State Field Problems	205
4.9	Convergence	207

5

Shape Functions of C_0 Continuity. Isoparametric Elements.....		211
--	--	------------

5.1	Introduction	211
5.2	Normalized Coordinates	212
5.3	Families of Shape Functions of C_0 Continuity	217
5.4	Transformations	229
5.5	Isoparametric Elements. Numerical Integration	238

6

Thin Plate According To Kirchhoff Theory 259

- 6.1 Introduction. Starting Hypothesis 259
- 6.2 Differential Approach. Field Equations. Definition of Stress 260
- 6.3 Integral Approach. Principle of Virtual Work 264
- 6.4 Finite Element Discretization 265
- 6.5 Requirement for continuity C_1 268
- 6.6 Nonconforming Plate Element 269
- 6.7 Conforming Plate Element 274
- 6.8 Conclusions 278

7

Thick plates. Reissner-Mindlin Theory 281

- 7.1 Introduction. 281
- 7.2 Differential Approach. Stress Definition 281
- 7.3 Integral Approach 285
- 7.4 Finite Element Discretization 286
- 7.5 Shear Locking Solution 287
- 7.6 Triangular Elements 297

A

Stiffness Matrices for Elements 301

- A.1 Elementary stiffness matrices 301

B

Loads on Elements 307

- B.1 Equivalent Nodal Loads 307

C

Rigid Nodes of Finite Size 315

D

Plane Stress and Strain 317

D.1 Plane Stress (XY Plane) 317

D.2 Plane Strain (XY Plane) 318

E

Numerical Integration 319

E.1 Quadrature Rules 319

E.2 Gaussian Quadrature 322

E.3 Multiple Integrals 325

F

Historical Reference 331

General Approach for the Finite Element Method. Application in Elastic and Steady-State Field Problems.

4 1

INTRODUCTION

The aim of this chapter is to present the general basis of the Finite Element Method (FEM). FEM is a way to obtain approximate solutions of partial differential equations. The approach is general, even though initially we will use the particular example of structural analysis to understand how the method is formulated.

For this purpose, we will first set forth the integral formulation of the boundary value problem, and subsequently we will introduce the concept of approximation. Then we will detail the characteristics that differentiate FEM from other methods. Once this general introduction is complete, we can deal with issues more focused on FEM: the concept of an element will be introduced and we will show how to synthesize global properties. The intention is to give the reader an overall understanding of the concepts that are characteristic of FEM.

After this discussion, we'll focus on linear plane elastic problems and steady state field problems.

4 2

DIFFERENTIAL APPROACH TO THE BOUNDARY VALUE PROBLEM

A boundary value problem is described by a mathematical model that approximates the solution in an acceptable way, usually in terms of

- equations in the domain

$$\mathbf{A}\mathbf{u} = \mathbf{f} \quad \text{in } \Omega \quad (4.1)$$

- and certain conditions (i.e., other equations) on its boundary

$$\mathbf{C}\mathbf{u} = \mathbf{g} \quad \text{en } \partial\Omega \quad (4.2)$$

where \mathbf{A} is a differential operator of order $2k$, and \mathbf{C} another differential operator on the boundary. Both operators are characteristic of boundary value problems (in the most general case using vector functions, these operators are matrices of partial derivatives) \mathbf{u} is the field variable, and may be a scalar or a vector quantity composed of a basis of several variables (in the Navier approach it is the displacement function vector). The value of \mathbf{u} depends on the location (space) and the instant we are studying (time). Finally, \mathbf{f} and \mathbf{g} are known functions which constitute the problem data and they are consistent with the operators. The definition spaces for \mathbf{u} and \mathbf{f} are \mathcal{U} and \mathcal{F} which, in general, are *metric spaces*, or spaces where the *distances between functions* are defined. Metric spaces allow us to determine if the functions are the same (if the *distance* between them is zero) or not, as well as to evaluate the relative error of two approximate solutions. Thus \mathbf{A} is an operator that transforms elements of \mathcal{U} into elements of \mathcal{F} and can be written as

$$\mathbf{A} : \mathcal{U} \rightarrow \mathcal{F} \quad (4.3)$$

This differential approach, based on the analysis of a differential element of the continuum, leads to a set of differential equations. It often requires that excessive restrictions be enforced on the variable field (it must be differentiable up to order $2k$ as stated above): it may be impossible to find a function that meets these derivability requirements.

On the other hand, the formulation of equations that govern boundary value problems also allows an integral approach. There are two alternatives:

- a minimization of a functional by a variational process, and
- a weak formulation (though this alternative encompasses the first).

The integral approach allows solutions with a lower degree of regularity, i.e., it reduces the demands on the solution, which as we will see later requires differentiable functions only up to order k . It is also of interest to note that when the differential equation has a solution, the *strong* or differential solution coincides with the *weak* solution. Conversely, there may be a solution using a weak approach, which means that the scope of solutions must be widened.

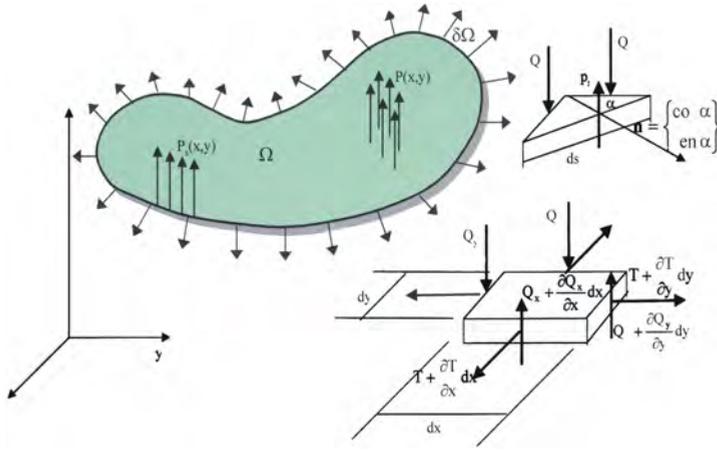


Figure 4E1.1

Example 4.1

Study the behavior of a inextensible and totally flexible membrane located in the xy plane subjected to loads in the z -axis direction. Assume as well that the membrane stresses in its plane do not depend on the section in which they are acting, i.e.,

$$T_x = T_y = T \tag{4E1.1}$$

The vertical components are denoted by Q_x and Q_y as can be seen in Figure 4E1.1.

Solution:

Establishing equilibrium in the x and y directions

$$\begin{cases} \frac{\partial T}{\partial x} = 0 \\ \frac{\partial T}{\partial y} = 0 \end{cases} \Rightarrow T = cte \tag{4E1.2}$$

as well as the z direction

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0 \tag{4E1.3}$$

on the boundary (see figure 4E1.1), equilibrium leads to

$$-Q_x \cos \alpha ds - Q_y \sin \alpha ds - p_s ds = 0 \tag{4E1.4}$$

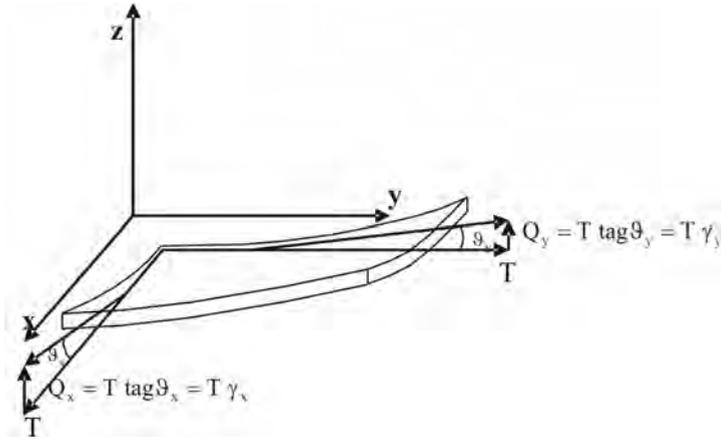


Figure 4E1.2

from which

$$p_s = Q_x \cos \alpha + Q_y \sin \alpha \tag{4E1.5}$$

Equations 4E1.3 and 4E1.5 can be written more compactly as

$$\begin{cases} \nabla^T \mathbf{Q} + \mathbf{p} = 0 & \text{in } \Omega \\ \mathbf{n}^T \mathbf{Q} + p_s = 0 & \text{in } \partial\Omega \end{cases} \tag{4E1.6}$$

The compatibility equations in domain Ω relate displacements w in direction z as well as the slopes:

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_x \\ \gamma_y \end{pmatrix} \tag{4E1.7}$$

so that

$$\boldsymbol{\gamma} = \nabla w \tag{4E1.8}$$

and on the boundary

$$w = w_g \tag{4E1.9}$$

Constitutive equations relate stress to deformations, and in this case (see Figure 4E1.2), we have:

$$\mathbf{Q} = T \boldsymbol{\gamma} \tag{4E1.10}$$

with the same meaning for each component.

When the compatibility equations are inserted into constitutive relations, and the result is inserted into the equilibrium equations, we have

$$\nabla^T (T \nabla w) + \mathbf{p} = 0 \quad \text{in } \Omega \tag{4E1.11}$$

This is the field equation of the problem in question, to which we will add boundary conditions, which include

- displacement (Dirichlet or essential)

$$w = w_s \quad \text{in} \quad \partial\bar{\Omega} \quad (4E1.12)$$

which will henceforth be considered homogenous, or $w = w_s = 0$ in this case and

- force (Neumann or natural)

$$\mathbf{n}^T \mathbf{Q}_s = p_s \quad \text{in} \quad \partial\bar{\Omega} \quad (4E1.13)$$

where $\partial\Omega = \partial\bar{\Omega} \cup \partial\bar{\bar{\Omega}}$

4.3

INTEGRAL APPROACH TO THE BOUNDARY VALUE PROBLEM

The weak approach consists in taking the dot product of two members of the differential equation (4.1) with a known function $\Psi \in C_n^k(\Omega)$. The resulting integration in the domain is

$$\int_{\Omega} \Psi^T \mathbf{A} \mathbf{u} d\Omega = \int_{\Omega} \Psi^T \mathbf{f} d\Omega \quad (4.4)$$

The first member of the equation (4.4) can now be integrated by parts k times (recalling that operator A includes derivatives up to order $2k$). For the linear case, which is this problem, we obtain the weak form

$$\int_{\Omega} (\mathbf{H}\Psi)^T (\mathbf{S}\mathbf{u}) d\Omega - \int_{\partial\Omega} (\mathbf{F}\Psi)^T (\mathbf{G}\mathbf{u}) d\Omega = \int_{\Omega} \Psi^T \mathbf{f} d\Omega \quad (4.5)$$

where \mathbf{H} , \mathbf{S} , \mathbf{F} and \mathbf{G} are linear differential operators, the first two having order k , \mathbf{G} contains derivatives from the order (k) to the order $(2k - 1)$ and \mathbf{F} is of order $(k - 1)$.

The previous process shows that the solution for (4.1) is the one where the projection of $\mathbf{A}\mathbf{u}$ on \mathcal{U} is equal to the projection of \mathbf{f} on \mathcal{U} , assuming it satisfies the demands of existence and uniqueness (Lax-Milgram theorem [44]).

This means that for every $\Psi \in \mathcal{U}$ function (as we can see, the projection space coincides with space \mathcal{U} where the field variable \mathbf{u} belongs), we can establish

$$\langle \mathbf{A}\mathbf{u}, \Psi \rangle = \langle \mathbf{f}, \Psi \rangle \quad (4.6)$$

which is identical to (4.4) with a different notation, in which

$$\langle a, b \rangle = \int_{\Omega} a b d\Omega$$

means *energetic* inner product. To make this possible, a and b must be bounded, so their integral makes sense as well as their square (so that $\langle \mathbf{f}, \mathbf{f} \rangle$ makes sense). In other words, these must be *square-integrable functions* in Ω , that is to say, $L_2(\Omega)$.

The subsequent integration by parts of (4.6) leads to expression (4.5) and a realization that \mathcal{U} must be a space whose functions are differentiable up to order k (the order of operator \mathbf{A} is $2k$). In addition, these derivatives must be square-integrable functions. A space with these characteristics is called a *Sobolev space of integer k* , $H_n^k(\Omega)$, with $n = 2$.

The second term of the integral equation (4.5) includes the Dirichlet (or first-type) and the natural (Neumann or second-type) boundary conditions, i.e. $\partial\Omega = \partial\Omega_N \cup \partial\Omega_D$ and $\emptyset = \partial\Omega_N \cap \partial\Omega_D$, where subscripts refer to the type of condition (Dirichlet or Neumann). To simplify the Dirichlet boundary conditions, we can extract them from the second term of (4.5) by making sure that the chosen functions Ψ are zero at the corresponding $\partial\Omega_D$ boundary. This integral formulation can also be achieved by applying physical concepts such as energy and variational calculus, as will now be explained

A *variational principle* specifies a scalar quantity (a functional) Π , defined by an integral expression

$$\Pi = \int_{\Omega} \mathbf{B}(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \dots) d\Omega + \int_{\partial\Omega} \mathbf{D}(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial \mathbf{x}}, \dots) d\Gamma \quad (4.7)$$

where \mathbf{u} is the unknown function and \mathbf{B} , \mathbf{D} , are specified operators.

The solution to the continuum problem is a function \mathbf{u} , which makes the functional Π stationary with respect to small δu variations. Thus the variation is

$$\delta\Pi = 0 \Rightarrow \int_{\Omega} \delta \mathbf{u}^T \mathbf{L}(\mathbf{u}) d\Omega + \int_{\partial\Omega} \delta \mathbf{u}^T \mathbf{M}(\mathbf{u}) d\Gamma \quad (4.8)$$

and this expression must be true for any small variation of \mathbf{u} (δu). Equivalently

$$\mathbf{L}(\mathbf{u}) = 0; \quad \mathbf{M}(\mathbf{u}) = 0 \quad (4.9)$$

where $\mathbf{M}(\mathbf{u})$ and $\mathbf{L}(\mathbf{u})$, may or may not be the differential equations (4.1) that govern the problem with the boundary conditions (4.2). In the general case the variational principle is called *natural* and the equations of (4.8) are known as differential equations of *Euler's variational principle*.

It is easy to show that for any variational principle we can establish a corresponding system of Euler equations, but unfortunately the inverse does not hold true and only certain forms of differential equations are Euler equations of a variational functional.

In the case of self-adjoint linear differential equations we can establish natural variational principles in a relatively simple way.

Thus, when the differential equation is

$$\mathbf{L}\mathbf{u} = \mathbf{A}\mathbf{u} + \mathbf{f} = 0; \quad (4.10)$$

and the self-adjoint or symmetric \mathbf{A} operator means

$$\int_{\Omega} \Psi^T \mathbf{A}(\mathbf{u}) d\Omega = \int_{\Omega} \mathbf{u}^T \mathbf{L}\Psi d\Omega \quad (4.11)$$

the variational principle can be written as

$$\Pi = \int_{\Omega} \left[\frac{1}{2} \mathbf{u}^T \mathbf{A}(\mathbf{u}) + \mathbf{u}^T \mathbf{f} \right] d\Omega + \text{boundary terms} \quad (4.12)$$

where we will consider the homogeneous Dirichlet boundary conditions. The natural or Neumann boundary conditions are not included, though these can be added easily. To verify that you can set these variational principles, consider the variation

$$\begin{aligned} \delta\Pi &= \int_{\Omega} \left[\frac{1}{2} \delta \mathbf{u}^T \mathbf{A}(\mathbf{u}) + \frac{1}{2} \mathbf{u}^T \delta(\mathbf{A}\mathbf{u}) + \delta \mathbf{u}^T \mathbf{f} \right] d\Omega \\ &\Rightarrow \int_{\Omega} \left[\frac{1}{2} \delta \mathbf{u}^T \mathbf{A}(\mathbf{u}) + \frac{1}{2} \mathbf{u}^T \mathbf{A} \delta \mathbf{u} + \delta \mathbf{u}^T \mathbf{f} \right] d\Omega \\ \delta\Pi &= \int_{\Omega} [\delta \mathbf{u}^T (\mathbf{A}(\mathbf{u}) + \mathbf{f})] d\Omega \quad (4.13) \end{aligned}$$

which proves the point.

Finally, there is a further interpretation of (4.5) which applies only to function Ψ as a function of virtual displacements, given the solitary restriction that it meet the Dirichlet boundary condition. In the elastic case, the terms $\mathbf{H}\Psi$ and $\mathbf{F}\Psi$ acquire the physical meaning of deformation and displacement functions corresponding to a virtual state (expressed as the superscript Ψ). This can be written as

$$\int_{\Omega} \epsilon^{\Psi T} \boldsymbol{\sigma} d\Omega - \int_{\partial\Omega} \mathbf{v}^{\Psi T} \mathbf{T} d\Gamma = \int_{\Omega} \Psi^T \mathbf{f} d\Omega \quad (4.14)$$

This is nothing more than a formulation of the *Principle of Virtual Work (PVW)*, where the virtual work of the external forces of a system in equilibrium ($\boldsymbol{\sigma}$, \mathbf{f} , \mathbf{T}) on the compatible virtual displacements (Ψ , ϵ^T) (corresponding to the sum of the second member and the second term of the first member of (4.11)) is equal to the internal energy due to the virtual strains (the first term of (4.11)).

Example 4.2

Present the weak form in the case of a stretched membrane as outlined in the example in the previous section.

Solution:

You can either multiply

$$\nabla^T (T \nabla w) + \mathbf{p} = 0 \quad (4E2.1)$$

by the known function and integrate by parts, or replace the equilibrium equation with

its equivalent obtained through the Principle of Virtual Work, obtaining

$$\left\{ \begin{array}{l} \int_{\Omega} (\nabla \Psi)^T T \nabla w \, dx dy = \int_{\Omega} \Psi \mathbf{p} \, dx dy + \int_{\partial \Omega_N} \Psi \mathbf{p}_s \, ds \\ \forall \Psi / \Psi = 0 \quad \text{in} \quad \partial \Omega_D \\ \mathbf{w} = \bar{\mathbf{w}} \quad \text{in} \quad \partial \Omega_D \end{array} \right. \quad (4E2.2)$$

To ensure that the integrals above make sense, we must verify that the field variable w as well as the function Ψ belong to the space of square integrable functions:

$$\Psi \in H^1(\Omega) \Rightarrow \int_{\Omega} [(\frac{\partial \Psi}{\partial x})^2 + (\frac{\partial \Psi}{\partial y})^2 + \Psi^2] \, dx dy < \infty \quad (4E2.3)$$

In the general case, when we have loads P_k or an elastic support (with stiffness K_k) in the domain, the integral equation is

$$\int_{\Omega} (\nabla \Psi)^T T \nabla w \, dx dy = \int_{\Omega} \Psi \mathbf{p} \, dx dy + \sum_{k=1}^n \Psi(x_k, y_k) [P_k - K_k w(x_k, y_k)] \quad (4E2.4)$$

4 4

APPROXIMATION

In the previous section the problem was formulated in integral terms, but early on we discovered that it is not easy to obtain an explicit solution by directly solving the original differential equation.

Instead, we can attempt find an approximate solution in a subspace U^N with finite dimension N from the original search space U , which has a *basis* function φ_i .

$$u \approx u^N = \sum_{i=1}^N \varphi_i a_i \quad (4.15)$$

To approximate the unknown function (field variable) with a linear combination of known functions, the problem is reduced to:

- defining the basis of functions φ_i , and
- obtaining the parameters, coefficients or generalized coordinates a_i of the approximate solution in the basis (4.15).

By substituting the approximate u^N (4.15) in the integral equation (4.5), and bearing in mind that integral and differential operators are linear, we can write

$$\sum_{i=1}^N a_i \int_{\Omega} (\mathbf{H}\Psi)^T (\mathbf{S}\varphi_i) \, d\Omega - \sum_{i=1}^N a_i \int_{\partial \Omega} (\mathbf{F}\Psi)^T (\mathbf{G}\varphi_i) \, d\Gamma = \int_{\Omega} \Psi^T \mathbf{f} \, d\Omega \quad (4.16)$$

an equation with N unknowns (values of $a_i, i = 1, \dots, N$), that may arise for N weighting functions or the projection $\psi_j \quad \forall j = 1, \dots, N$ linearly independent, leading to a system of N equations with N unknowns. If you match up the equations of this approach and the weighting functions $\psi_i = \varphi_i$, the result constitutes the Bubnov-Galerkin formulation. This system of equations can be written as

$$\sum_{i=1}^N a_i \int_{\Omega} (\mathbf{H}\varphi_j)^T (\mathbf{S}\varphi_i) d\Omega - \sum_{i=1}^N a_i \int_{\partial\Omega} (\mathbf{F}\varphi_j)^T (\mathbf{G}\varphi_i) d\Gamma = \int_{\Omega} \varphi_j \mathbf{f} d\Omega \quad \forall j = 1, \dots, N \quad (4.17)$$

and its solution will allow us to identify coefficients a_i .

Clearly, the projection space (basis Ψ_i) and the solution subspace (of basis φ_i) do not need to be equal. In fact, the definition of these spaces is one of the main differences between approximation methods.

When dealing with approximation, the values u^N do not have to satisfy the Dirichlet boundary conditions (u^N belongs to a vectorial subspace) except if they are zero. In this case, a solution can be found by assuming that there is a function u^0 that satisfies the boundary conditions, and then, instead of employing (4.15), using the following method

$$u - u^0 \approx u^N = \sum_{i=1}^N \varphi_i a_i \quad (4.18)$$

From now on, to simplify the formulation, we will only consider homogeneous Dirichlet boundary conditions.

Next, we will include a brief outline of different approximation methods, which will also help us place the theory in a broader context.

1. Weighted residual methods

These methods define the error or residual from (4.1), as

$$\mathbf{R}(u^N) = \mathbf{A}u^N - \mathbf{f} \quad (4.19)$$

The idea is to find coordinates a_i of the approximation u^N , defined in (4.15) that minimize error. This can be achieved through the norm

$$\|v\|^2 = \langle v, v \rangle \quad (4.20)$$

or setting the orthogonality of the error with respect to a subspace (defined by the base φ_i). This approach is the basis of

- *the Galerkin method*, where the residual is orthogonal to the subspace of the same dimension (N) where the approximation was made (see (4.15))

$$\langle \mathbf{R}(u^N), \psi_j \rangle = 0 \quad \forall j = 1, \dots, N \quad (4.21)$$

The Petrov-Galerkin method is used if the approximating and the weighting spaces are different: if they match, the Bubnov-Galerkin method (more commonly known as the Galerkin method) is used.

As can be seen, expression (4.21) is the starting point of the previously developed weak formulation. Note that in the Bubnov-Galerkin method the weighting conditions must comply with the Dirichlet homogeneous boundary conditions because the approximation requires it: in this case both the approximation and the weighting functions match.

- *the least squares method*, which minimizes the integral over the domain of the squared residual or norm (for vectorial functions $\|\mathbf{R}(\mathbf{u}^N)\|^2$), i.e.

$$\frac{\partial}{\partial a_i} \int_{\Omega} \|\mathbf{R}(\mathbf{u}^N)\|^2 d\Omega = 0 \quad (4.22)$$

Operating

$$\int_{\Omega} \mathbf{R}(\mathbf{u}^N) \frac{\partial \mathbf{R}(\mathbf{u}^N)}{\partial a_i} = 0 \quad (4.23)$$

which, as shown, is just the Petrov-Galerkin method, in which we have chosen as weighting functions $\varphi_i = \frac{\partial \mathbf{R}(\mathbf{u}^N)}{\partial a_i}$.

- *the point collocation method*, where we choose Dirac delta distributions the simplest of possible weighting functions:

$$\varphi_i = \delta_i \quad (4.24)$$

By definition of the direct delta distribution (2) the integral equation (4.21) is reduced by setting the residue to zero at specific x_i points in the domain, i.e.,

$$\mathbf{R}(\mathbf{u}^N) = 0, \quad \text{in } x_i \quad \forall i = 1, \dots, N \quad (4.25)$$

2. Ritz Method

This is the only variational method we will discuss (others are Treffe, Kantorovick, etc.) because of its equivalence with the Galerkin method.

For the Ritz method, the best method to obtain the system of equations is to find where the value of the functional Π is a minimum.

If we consider the case of self-adjoint operators, the variational principle can be written in the form of (4.12). Inserting the approximate value from (4.15), we have

$$\Pi = \int_{\Omega} \left[\frac{1}{2} \left(\sum_{i=1}^N \varphi_i a_i \right) \mathbf{A} \left(\sum_{i=1}^N \varphi_i a_i \right) - \left(\sum_{i=1}^N \varphi_i a_i \right) \mathbf{f} \right] d\Omega + t.c \quad (4.26)$$

and minimizing

$$\frac{\partial \Pi}{\partial a} = 0 = \frac{1}{2} \sum_{i=1}^N a_i \int_{\Omega} (\mathbf{A} \varphi_j) \varphi_i d\Omega - \sum_{i=1}^N \int_{\Omega} \mathbf{f} \varphi_i d\Omega + t.c \quad (4.27)$$

which is identical to (4.18) and (4.21) using the Bubnov-Galerkin method.

Example 4.3

Obtain the Galerkin approximation for the problem of a stretched membrane presented in Examples 4E1 and 4E2

Solution:

This involves inserting the approximate values w_N into the weak formulation. The approximate values of the field variable, in this case the displacement normal to the plane of the membrane, are obtained as a combination of N functions of a particular basis of functions

$$w_N(x, y) = \sum_i^N a_j \varphi_j(x, y) \quad (4E3.1)$$

Since the problem is being solved using a Galerkin approximation, the functions of this basis coincide with the weight functions ($\varphi_i = \psi_i$) thus

$$\sum_i^N a_j \left[\int_{\Omega} (\nabla \varphi_i)^T T \nabla \varphi_j dx dy + \sum_i^N \varphi_i(x_k, y_k) k_k \varphi_j(x_k, y_k) \right] = \int_{\Omega} \varphi_i p dx dy + \sum_i^N \varphi_i(x_k, y_k) P_k \quad (4E3.2)$$

which can be written as

$$\sum_i^N k_{ij} a_j = f_j \quad (4E3.3)$$

where

$$\begin{cases} k_{ij} = \int_{\Omega} (\nabla \varphi_i)^T T \nabla \varphi_j dx dy + \sum_i^N \varphi_i(x_k, y_k) k_k \varphi_j(x_k, y_k) \\ f_j = \int_{\Omega} \varphi_i p dx dy + \sum_i^N \varphi_i(x_k, y_k) P_k \end{cases} \quad (4E3.4)$$

4 5**THE FINITE ELEMENT METHOD****4 5 1 FEM characteristics**

Without a doubt, FEM is currently the most widely used method of approximation. It has all the salient features of the other approximation methods because it uses an integral formulation as well as a typical linear combination of functions. However, it possesses some essential characteristics that distinguish it. These are discussed below.