Contents

CHAPTER 1

Introduction. Classical Approximation

1.1 - INTRODUCTION

The aim of this chapter is to organize some generally known concepts in ways that are useful for analyzing stress and strain in bar structures. The first step is to establish the basic problem: what variables we want to know and what tools are available. When we express this problem in algebraic terms, we obtain an equal number of unknowns and equations.

In this first section we demonstrate how the basic equations of mechanics are more than adequate for structural analysis.

Then, we propose the stiffness method as a simple guideline for solving systems of equations, using a substitution methodology. The third section repeats the process using matrix notation, which is more convenient for computers. The final section summarizes our key findings.

To illustrate our theory with a practical application we will use a basic structure, likely well known to the readers: the hyperstatic lattice in Figure 1.1(a). The size and characteristics are indicated in the figure:

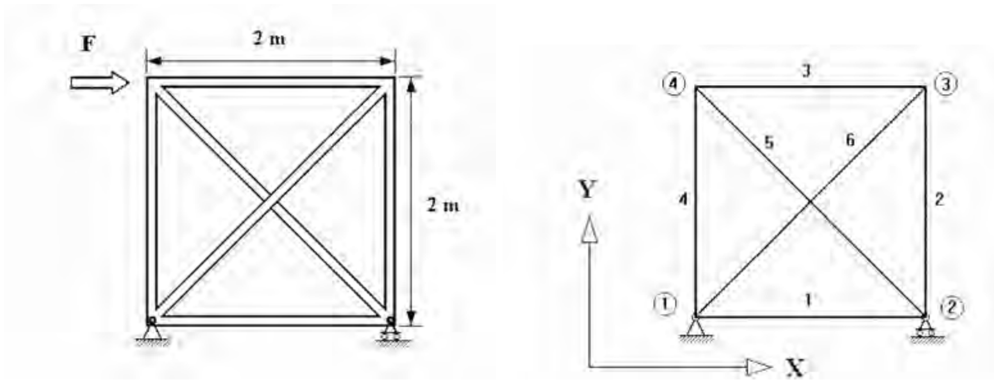
Area of the bars: 10 cm^2, elastic modulus: $2.1E6\,\text{kg/cm}^2$, applied load: $F = 5000\,\text{kg}$.

We have numbered each specific node or bar in the structure, as shown in figure 1.1(b), as well as a coordinate or axis system representing the directions and positive orientations, both as loads and displacements.

1.2 - POSING THE PROBLEM

This section explains the goals of this analysis as well as giving the necessary tools for a typical lattice calculation. When the problem is solved, the following quantities will be obtained:

Stress in each bar: since the only consideration is axial stress, it represents an unknown in each bar. It will be noted as Ni, indicating the bar number in the subscript. Positive tensile strength is assumed.



(a) (b)

Figure 1.1

Displacements of the nodes: horizontal and vertical directions, which means two unknowns per node, notated as u_i and v_i.

Support reactions: an unknown in each restricted direction (constraint) which will be notated as X_i, Y_i, depending on whether it is horizontal or vertical.

Sometimes, the above quantities are summarized in the following symbolic relationship expressed by the function below that considers the problem in two dimensions:

$$I = b + 2n + r$$

where:
I: number of unknowns
b: number of bars
n: number of nodes
r: number of constraints
In the proposed structure:

$$\left\{ \begin{array}{lll} b & = 6 & (N_1, N_2, N_3, N_4, N_5, N_6) \\ 2n & = 8 & (u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4) \\ r & = 3 & (X_1, Y_1, Y_2) \end{array} \right. \Rightarrow I = 17$$

The u_1, v_1 and v_2 constrained displacements at the supports are considered redundant, since it is already known that they are void.

Once the number of unknowns is established, the problem is finding enough equations for a system with a solution. The first set of these equations are the boundary conditions, which are given by the number r (one for each unknown reaction, since a reaction appears at the nodes where a single displacement is imposed). In the example structure:

$$u_1 = v_1 = v_2 = 0 \tag{1.1}$$

The force balance at the nodes gives two equations for each node. These equations link the stress in the elements to the externally applied loads at the node. Figure 1.2, shows the equations for node 4 of the structure.

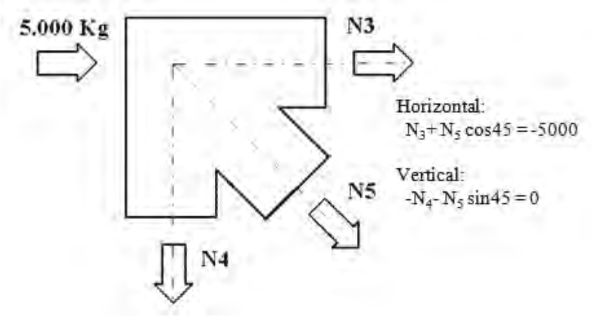


Figure 1.2

Usually, in order to avoid changing the sign of the loads from positive to negative (when 5000 kg is applied to node 4), the signs of all the equations are reversed. Then, for the entire structure the following is obtained:

$$\begin{cases} Node \quad 1: & -N_1 - N_6 \cos 45 = X_1 & ; & -N_4 - N_6 \sin 45 = Y_1 \\ Node \quad 2: & N_1 + N_5 \cos 45 = 0 & ; & -N_2 - N_5 \sin 45 = Y_2 \\ Node \quad 3: & N_3 + N_6 \cos 45 = 0 & ; & N_2 + N_6 \sin 45 = 0 \\ Node \quad 4: & -N_3 - N_5 \cos 45 = 5000 & ; & N_4 + N_5 \sin 45 = 0 \end{cases} \tag{1.2}$$

With the systems (1.1) and (1.2) $2n + r$ equations are derived. For the other b equations, two basic concepts from structural mechanics are used: *constitutive relations* and *compatibility*.

In the case of lattice elements, the behavior is expressed using a well-known relationship:

$$N_i = A_i E_i \frac{\Delta L_i}{L_i}$$

The direct application of this expression introduces a new unknown factor for each bar: the increase in length. This variable is of interest as it can represent the deformation of each element. In the most general case, it is also an intermediate variable calculation that can be eliminated by simply substituting the compatibility equations that relate the increase in length of each bar to the movements of its end nodes.

In the case of small displacements, obtaining an analytical expression for these equations is a question of basic geometry. Figure 1.3 shows this process.

$$\Delta L_k = U_{Lj} - U_{Li} = u_j \cos\alpha + v_j \sin\alpha - (u_i \cos\alpha + v_i \sin\alpha)$$

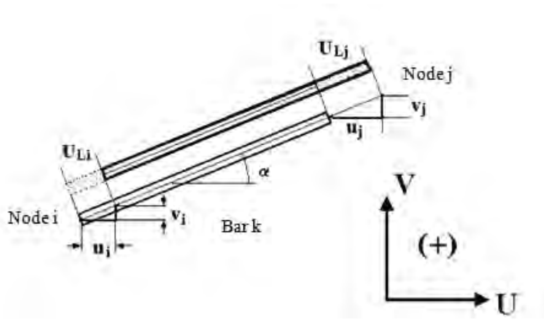


Figure 1.3

When the equations of behavioral compatibility are substituted into these equations, for each bar the following is obtained:

$$N_k = \frac{A_k E_k}{L_k}(u_j \cos\alpha + v_j \sin\alpha - u_i \cos\alpha - v_i \sin\alpha)$$

The b equations necessary to get the same number of equations and unknowns have been defined. The numerical values of the parameters A, E, L, α, when substituted in the b equations of the example structure, yield

$$\begin{cases} Bar \quad 1 : N_1 = 1.05E5(u_2 - u_1) \\ Bar \quad 2 : N_2 = 1.05E5(v_3 - v_2) \\ Bar \quad 3 : N_3 = 1.05E5(u_3 - u_4) \\ Bar \quad 4 : N_4 = 1.05E5(v_4 - v_1) \\ Bar \quad 5 : N_5 = \dfrac{1.05E5}{2}(u_2 - v_2 - u_4 + v_4) \\ Bar \quad 6 : N_6 = \dfrac{1.05E5}{2}(u_3 + v_3 - u_1 - v_1) \end{cases} \tag{1.3}$$

1.3 - THE DIRECT STIFFNESS (OR DISPLACEMENT) METHOD

When system (1.1) is replaced in (1.3) and then in (1.2) a set of $2n$ equations is obtained, in this case $2n = 8$, which represents the structure's nodal equilibrium. The equilibrium equations are the last to be substituted. The expression for the structure being studied is:

$$\begin{cases} -1.05E5u_2 - \dfrac{1.05E5}{2\sqrt{2}}(u_3 + v_3) = X_1 \\ -1.05E5v_4 - \dfrac{1.05E5}{2\sqrt{2}}(u_3 + v_3) = Y_1 \\ 1.05E5u_2 + \dfrac{1.05E5}{2\sqrt{2}}(u_2 - u_4 + v_4) = 0 \\ -1.05E5v_3 - \dfrac{1.05E5}{2\sqrt{2}}(u_2 - u_4 + v_4) = Y_2 \\ 1.05E5(u_3 - u_4) + \dfrac{1.05E5}{2\sqrt{2}}(u_3 + v_3) = 0 \\ 1.05E5v_3 + \dfrac{1.05E5}{2\sqrt{2}}(u_3 + v_3) = 0 \\ -1.05E5(u_3 - u_4) - \dfrac{1.05E5}{2\sqrt{2}}(u_2 - u_4 + v_4) = 5000 \\ 1.05E5v_4 + \dfrac{1.05E5}{2\sqrt{2}}(u_2 - u_4 + v_4) = 0 \end{cases} \tag{1.4}$$

An important aspect of this system is that those equations where support reactions do not appear can be extracted and solved separately. This is possible because the $2n - r$ equations, once solved, generate 2n-r unknowns (nodal displacements unimpeded). Considering the example, the system in matrix form is:

$$1.05E5 \begin{pmatrix} (1+\frac{1}{2\sqrt{2}}) & 0 & 0 & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ 0 & (1+\frac{1}{2\sqrt{2}}) & \frac{1}{2\sqrt{2}} & -1 & 0 \\ 0 & \frac{1}{2\sqrt{2}} & (1+\frac{1}{2\sqrt{2}}) & 0 & 0 \\ -\frac{1}{2\sqrt{2}} & -1 & 0 & (1+\frac{1}{2\sqrt{2}}) & -\frac{1}{2\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & 0 & 0 & -\frac{1}{2\sqrt{2}} & (1+\frac{1}{2\sqrt{2}}) \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5000 \\ 0 \end{pmatrix} \tag{1.5}$$

After solving the system, it is obtained:

$$\begin{pmatrix} u_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0.02381 \\ 0.09115 \\ -0.02381 \\ 0.11496 \\ 0.02381 \end{pmatrix} cm$$

Once the displacements are known, the other variables can be found very easily. The reactions can also be obtained by substituting these displacements in the r equilibrium equations (in this case $r = 3$) for the supports which were set aside when moving from system (1.4) to system (1.5). As a result, the following is obtained

$$\begin{pmatrix} X_1 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} -5000 \\ -5000 \\ 5000 \end{pmatrix} kg$$

Obtaining the stress in the bar only requires the replacement of the displacements in system (1.3). For instance, for bar 5 of the structure it results

$$N_5 = \frac{1,05E5}{2}(u_2 - v_2 - u_4 + v_4) = -3535,35 \quad kg.$$

1.4 - MATRIX FORMULATION

The previous section ended up with the matrix expression of the system but either formulation can be used. Which one to choose is, in principle, irrelevant. However, when analyzing a structure it becomes an important consideration because this choice designates the solution method.

First represent the equilibrium at the nodes with the following matrix notation:

$$\begin{pmatrix} F_{ix} \\ F_{iy} \end{pmatrix} + \sum_k \begin{pmatrix} Q_{ix}^k \\ Q_{iy}^k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This last equation states that the external loads applied to node i plus the sum of the actions that are exerted on the node by all the bars k have a zero result. The subscripts x and y refer to the direction in which the balance is considered. The equations shown before can now be expressed as

$$\begin{Bmatrix} F_{ix} \\ F_{iy} \end{Bmatrix} = -\sum_k \boldsymbol{Q}_i^k = \sum_k \boldsymbol{S}_i^k$$

where the quantity Q_i^k represents the action of the bar on the node and its inverse S_i^k (the reaction of the node on the bar).

$$\boldsymbol{Q}_i^k = \begin{Bmatrix} Q_{ix}^k \\ Q_{iy}^k \end{Bmatrix}, \quad \boldsymbol{S}_i^k = \begin{Bmatrix} S_{ix}^k \\ S_{iy}^k \end{Bmatrix}$$

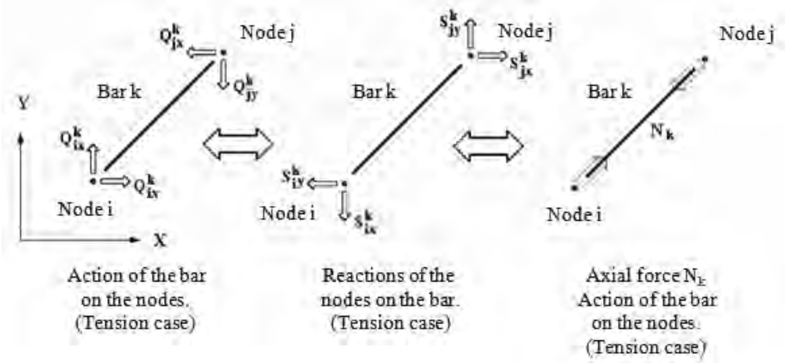


Figure 1.4

In the case of this lattice in two dimensions, it is

$$\begin{cases} Node \quad 2: & 0 = S^1_{2x} + S^2_{2x} + S^5_{2x} \\ Node \quad 3: & \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} S^2_{3x} \\ S^2_{3y} \end{Bmatrix} + \begin{Bmatrix} S^3_{3x} \\ S^3_{3y} \end{Bmatrix} + \begin{Bmatrix} S^6_{3x} \\ S^6_{3y} \end{Bmatrix} \\ Node \quad 4: & \begin{Bmatrix} 5000 \\ 0 \end{Bmatrix} = \begin{Bmatrix} S^3_{4x} \\ S^3_{4y} \end{Bmatrix} + \begin{Bmatrix} S^4_{4x} \\ S^4_{4y} \end{Bmatrix} + \begin{Bmatrix} S^5_{4x} \\ S^5_{4y} \end{Bmatrix} \end{cases}$$

and the resulting system of equations is

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 5000 \\ 0 \end{Bmatrix} = \begin{Bmatrix} S^1_{2x} + S^2_{2x} + S^5_{2x} \\ \begin{Bmatrix} S^2_{3x} \\ S^2_{3y} \end{Bmatrix} + \begin{Bmatrix} S^3_{3x} \\ S^3_{3y} \end{Bmatrix} + \begin{Bmatrix} S^6_{3x} \\ S^6_{3y} \end{Bmatrix} \\ \begin{Bmatrix} S^3_{4x} \\ S^3_{4y} \end{Bmatrix} + \begin{Bmatrix} S^4_{4x} \\ S^4_{4y} \end{Bmatrix} + \begin{Bmatrix} S^5_{4x} \\ S^5_{4y} \end{Bmatrix} \end{Bmatrix} \tag{1.6}$$

In the matrix equation (1.6) the equations that are related to the reactions have been eliminated.

The next step is to obtain a relationship between the actions S^k_i and the nodal displacements. By following a parallel process and using the theory illustrated in the previous sections, the concepts of compatibility and constitutive relations can be applied. Then, for any bar k with end nodes i and j (Figure 1.4) the following expression is obtained:

$$\left\{\begin{array}{c} S^k_{ix} \\ S^k_{iy} \\ S^k_{jx} \\ S^k_{jy} \end{array}\right\} = \left\{\begin{array}{c} -N_k \cos\alpha \\ -N_k \sin\alpha \\ N_k \cos\alpha \\ N_k \sin\alpha \end{array}\right\} = N_k \left\{\begin{array}{c} -\cos\alpha \\ -\sin\alpha \\ \cos\alpha \\ \sin\alpha \end{array}\right\}$$

But

$$N_k = \frac{A_k E_k}{L_k}(u_j \cos\alpha + v_j \sin\alpha - u_i \cos\alpha - v_i \sin\alpha)$$

Therefore

$$\left\{\begin{array}{c} S^k_{ix} \\ S^k_{iy} \\ S^k_{jx} \\ S^k_{jy} \end{array}\right\} = \frac{A_k E_k}{L_k} \left(\begin{array}{cc|cc} \cos^2\alpha & \cos\alpha\sin\alpha & -\cos^2\alpha & -\cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha & -\cos\alpha\sin\alpha & -\sin^2\alpha \\ \hline -\cos^2\alpha & -\cos\alpha\sin\alpha & \cos^2\alpha & \cos\alpha\sin\alpha \\ -\cos\alpha\sin\alpha & -\sin^2\alpha & \cos\alpha\sin\alpha & \sin^2\alpha \end{array}\right) \left\{\begin{array}{c} u_i \\ v_i \\ u_j \\ v_j \end{array}\right\}$$

The matrix relation above can be expressed in terms of the sections delimited by the dashed lines, as follows:

$$\begin{pmatrix} \boldsymbol{S^k_i} \\ \boldsymbol{S^k_j} \end{pmatrix} = \begin{pmatrix} \boldsymbol{K^k_{ii}} & \boldsymbol{K^k_{ij}} \\ \boldsymbol{K^k_{ji}} & \boldsymbol{K^k_{jj}} \end{pmatrix} \begin{pmatrix} \boldsymbol{u_i} \\ \boldsymbol{u_j} \end{pmatrix} \tag{1.7}$$

This definition will be shown to be very useful. Usually this relationship is written as

$$\boldsymbol{S} = \boldsymbol{K_e u}$$

K_e is called the *elemental stiffness matrix*. This matrix has a very intuitive interpretation. One can imagine a displacement vector

$$\boldsymbol{u} = \left\{\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right\}$$

This vector corresponds to a horizontal unitary displacement at node i, while all others are held at a displacement of zero. When multiplied by the stiffness matrix, a force vector is obtained:

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} & K_{11} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{pmatrix} \left\{\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right\} = \left\{\begin{array}{c} K_{11} \\ K_{21} \\ K_{31} \\ K_{41} \end{array}\right\}$$

These forces are the actions on the bar at its ends. One can think of each column of the bar stiffness matrix as the load vector that appears when a unitary displacement is given in the corresponding direction, while holding the other displacements equal to zero.

For the example structure, it is:

$$\begin{Bmatrix} S^1_{1x} \\ S^1_{1y} \\ S^1_{2x} \\ S^1_{2y} \end{Bmatrix} = 1.05E5 \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\begin{Bmatrix} S^2_{2x} \\ S^2_{2y} \\ S^2_{3x} \\ S^2_{3y} \end{Bmatrix} = 1.05E5 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} S^3_{3x} \\ S^3_{3y} \\ S^3_{4x} \\ S^3_{4y} \end{Bmatrix} = 1.05E5 \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\begin{Bmatrix} S^4_{1x} \\ S^4_{1y} \\ S^4_{4x} \\ S^4_{4y} \end{Bmatrix} = 1.05E5 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\begin{Bmatrix} S^5_{2x} \\ S^5_{2y} \\ S^5_{4x} \\ S^5_{4y} \end{Bmatrix} = \frac{1.05E5}{2\sqrt{2}} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\begin{Bmatrix} S^6_{1x} \\ S^6_{1y} \\ S^6_{3x} \\ S^6_{3y} \end{Bmatrix} = \frac{1.05E5}{2\sqrt{2}} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix}$$

Substituting the last six systems of linear equations into equation (1.6) and taking into account the boundary conditions, the following is obtained

$$\{0\} = 1.05E5\{1\}\{u_2\} + 1.05E5\left\{\begin{matrix} 0 & 0 & 0 \end{matrix}\right\}\left\{\begin{matrix} u_2 \\ u_3 \\ v_3 \end{matrix}\right\} +$$

$$\frac{1.05E5}{2\sqrt{2}}\left\{\begin{matrix} 1 & -1 & 1 \end{matrix}\right\}\left\{\begin{matrix} u_2 \\ u_4 \\ v_4 \end{matrix}\right\}$$

$$\left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\} = 1.05E5\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left\{\begin{matrix} u_2 \\ u_3 \\ v_3 \end{matrix}\right\} + 1.05E5\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\left\{\begin{matrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix}\right\}$$

$$+\frac{1.05E5}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\left\{\begin{matrix} u_3 \\ v_3 \end{matrix}\right\}$$

$$\left\{\begin{matrix} 5000 \\ 0 \end{matrix}\right\} = 1.05E5\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\left\{\begin{matrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{matrix}\right\} + 1.05E5\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\left\{\begin{matrix} u_4 \\ v_4 \end{matrix}\right\}$$

$$+\frac{1.05E5}{2\sqrt{2}}\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\left\{\begin{matrix} u_2 \\ u_4 \\ v_4 \end{matrix}\right\}$$

When the above equations are ordered, a system whose matrix expression is that given in (1.5) can be obtained. Formally, it is written as

$$\boldsymbol{F} = \boldsymbol{K}\,\boldsymbol{u}$$

$\boldsymbol{F}$ represents the vector of applied loads at the nodes, $\boldsymbol{u}$ the vector of displacements that occur at the nodes, and $\boldsymbol{K}$ the coefficient matrix of the system, known as the stiffness matrix. Some of the properties of this matrix, which will be studied in more detail in subsequent chapters, are *symmetry* and that all of the elements of the main diagonal are positive.

The process of substituting the bar equations (1.7) into the global equilibrium equations (1.6) can be organized into an algorithm known as *assembly*, which will be explained in detail later.

1.5 - CONCLUSIONS

The reader should not be discouraged if the first question that is being asked (*how does a computer evaluate structural equations?* goes unanswered for the time being. Everything that has been shown so far is the basis of every simple matrix calculation code. However, in order to build these codes, some knowledge of programming tools is required. It unnecessarily complicates things to explain how to do this at this juncture.

The reader should not be worried about the limited scope of a 2-D structural analysis of a lattice. All of these concepts are quite general and the analytical expressions can be different for each case. By carefully studying the exercises presented at the end of the unit, it will be possible to see how these ideas can be applied to several structural elements.

After reading this first chapter, the reader should understand that the analysis of bar structures with matrix calculations does not require any new concepts. This approach is, unfortunately, limited in scope. For example, if a structure has two- and three-dimensional elements, the terms *bar* and *node* become ambiguous, leaving the analyst helpless. It is needed to seek new, more powerful and more general insights.

All this will be discussed in the following chapters.

1.6 - APPLICATION EXAMPLES

Example 1.1
Development of the stiffness matrix for the bars of a plane frame.

Solution:

To solve this exercise it is required to analyze the relationship between displacement and stress for the most general case of a two-dimensional bar structure. It is assumed that this bar is capable of transmitting bending moments (as opposed to lattice elements) and is in axial strain (unlike in the classical analysis of frames).

It is necessary to relate the six stresses and strains as indicated in Figure 1E1.1.

Begin by expressing the relationship in a convenient coordinate system. Here, the most appropriate coordinate system is defined by the axial and normal directions of the bar (Figure 1E1.2). These axes are often labelled *local* because they only refer to the bar in question. In local coordinates, the relationship can be expressed as

$$\boldsymbol{S} = \boldsymbol{K_e}\boldsymbol{\delta}$$

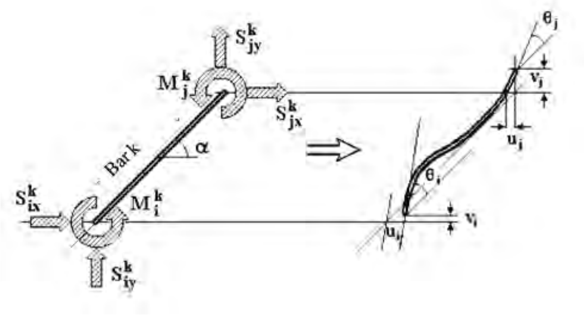


Figure 1E1.1

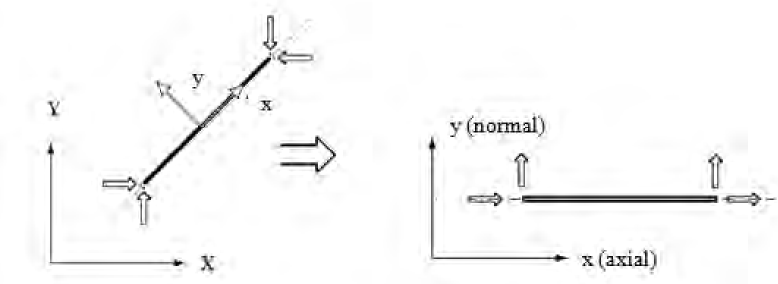


Figure 1E1.2

To obtain the terms of the stiffness matrix $\boldsymbol{K_e}$, give unitary displacements to successive bar ends holding all the others fixed.

As an example, in Figure 1E1.3 the calculations for the first three columns are developed.

Applying symmetries it is very easy to calculate the remaining terms. The result

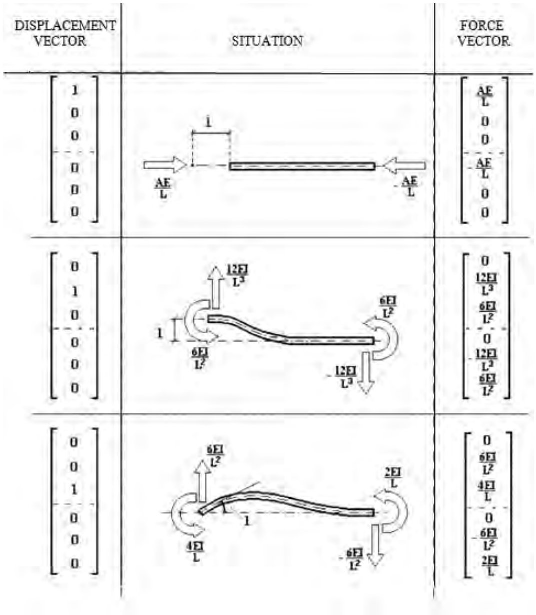


Figure 1E1.3

is a stiffness matrix

$$\boldsymbol{K}_e = \begin{pmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{pmatrix} \tag{1E1.1}$$

The problem is therefore solved. This matrix can be expressed in any coordinate system (see 1E1.1). It is only necessary to establish the relationship between the forces and displacements. According to Figure 1E1.4, this relationship (represented in matrix form) is:

$$\boldsymbol{\delta} = \boldsymbol{L}^T \boldsymbol{u}$$

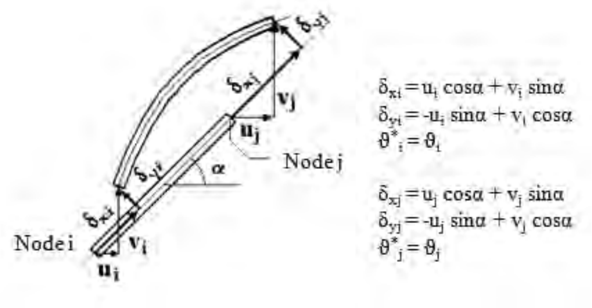


Figure 1E1.4

where

$$\boldsymbol{\delta} = \begin{Bmatrix} \delta_{xi} \\ \delta_{yi} \\ \theta_i^* \\ \delta_{xj} \\ \delta_{yj} \\ \theta_j^* \end{Bmatrix}, \quad \boldsymbol{L}^T = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 & 0 & 0 & 0 \\ -\sin\alpha & \cos\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{u} = \begin{Bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{Bmatrix}$$

The coordinate changing matrix $\boldsymbol{L}$ also includes stress. Then it can be written as

$$\boldsymbol{S}^* = \boldsymbol{L}^T \boldsymbol{S} \tag{1E1.2}$$

This coordinate changing matrix has the property that both its transpose and its inverse are identical.

Turning to the relationship between forces and displacements expressed in local coordinates and making the change of coordinates, the following is obtained

$$\boldsymbol{S}^* = \boldsymbol{K_e}^* \boldsymbol{\delta} \Rightarrow \boldsymbol{L}^T \boldsymbol{S} = \boldsymbol{K_e}^* \boldsymbol{L}^T \boldsymbol{u} \Rightarrow \boldsymbol{S} = \boldsymbol{L} \boldsymbol{K_e}^* \boldsymbol{L^T} \boldsymbol{u} \tag{1E1.3}$$

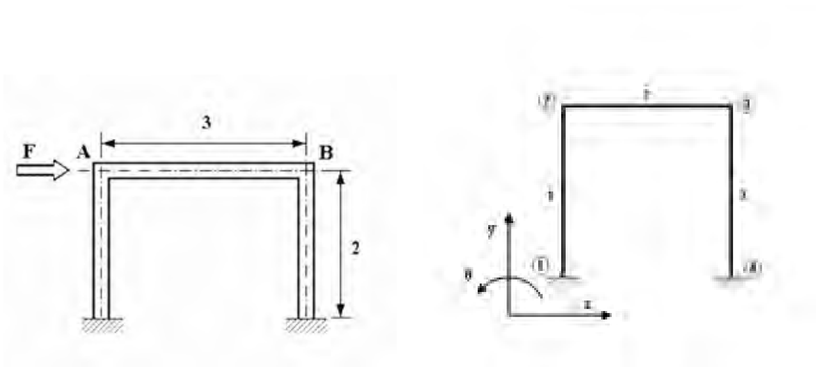
Therefore, the matrix for the most general case would be

$$\boldsymbol{K_e} = \boldsymbol{L} \boldsymbol{K_e}^* \boldsymbol{L}^T$$

Example 1.2
Find the displacements at points A and B of the structure in Figure 1E2.1(a), given the following information: Cross-sectional area of bars $= 900\,cm^2$; moment inertia of bars =8.0E5 cm^4; material modulus of elasticity = 2.0E5 kg/cm^2; applied load $F = 10000\,kg$.

Solution:

As with the above mentioned plane truss, the first step is to identify bars and nodes by their number as seen in Figure 1E2.1(b). Then, identify the unknowns and the equations of the problem. The unknowns are the nodal displacements and the equations are the equilibrium equations for the same nodes.



(a) (b)

Figure 1E2.1

Building on the previous exercise, the calculation can begin by obtaining the stiffness matrix for each bar (given in Equation (1E1.1)), which relates the forces and displacements at the ends as in Equation (1E1.3). The displacements and the stress at the end of each bar should have the same orientation so that there are no difficulties using the equilibrium equations. Then, if positive directions are chosen for displacement and rotation in each node x, y, θ in Figure 1E2.1(b), the angles to use for the change of coordinates matrix $\boldsymbol{L}$ are: $\alpha = 90$, $\alpha = 0$ and $\alpha = -90$ for bars 1, 2 and 3, respectively. When it is taken into account (1E1.2), the following is obtained

Bar 1

$$
\left(\begin{array}{c} S^1_{1x} \\ S^1_{1y} \\ M^1_1 \\ \hline S^1_{2x} \\ S^1_{2y} \\ M^1_2 \end{array}\right)
=
\left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{ccc|ccc} 9E5 & 0 & 0 & -9E5 & 0 & 0 \\ 0 & 2.4E5 & 2.4E7 & 0 & -2.4E5 & 2.4E7 \\ 0 & 2.4E7 & 3.2E9 & 0 & -2.4E7 & 1.6E9 \\ \hline -9E5 & 0 & 0 & 9E5 & 0 & 0 \\ 0 & -2.4E5 & -2.4E7 & 0 & 2.4E5 & -2.4E7 \\ 0 & 2.4E7 & 1.6E9 & 0 & -2.4E7 & 3.2E9 \end{array}\right)
$$

$$
\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{c} u_1 \\ v_1 \\ \theta_1 \\ \hline u_2 \\ v_2 \\ \theta_2 \end{array}\right)
=
\left(\begin{array}{ccc|ccc} 2.4E5 & 0 & -2.4E7 & -2.4E5 & 0 & -2.4E7 \\ 0 & 9E5 & 0 & 0 & -9E5 & 0 \\ -2.4E7 & 0 & 3.2E9 & 2.4E7 & 0 & 1.6E9 \\ \hline -2.4E5 & 0 & 2.4E7 & 2.4E5 & 0 & 2.4E7 \\ 0 & -9E5 & 0 & 0 & 9E5 & 0 \\ -2.4E7 & 0 & 1.6E9 & 2.4E7 & 0 & 3.2E9 \end{array}\right)
\left(\begin{array}{c} u_1 \\ v_1 \\ \theta_1 \\ \hline u_2 \\ v_2 \\ \theta_2 \end{array}\right)
\tag{1E2.1}
$$

Bar 2

$$
\left(\begin{array}{c} S^2_{2x} \\ S^2_{2y} \\ M^2_2 \\ \hline S^2_{3x} \\ S^2_{3y} \\ M^2_3 \end{array}\right)
=
\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{ccc|ccc} 6E5 & 0 & 0 & -6E5 & 0 & 0 \\ 0 & 7.11E4 & 1.07E7 & 0 & -7.11E4 & 1.07E7 \\ 0 & 1.07E7 & 2.13E9 & 0 & -1.07E7 & 1.07E9 \\ \hline -6E5 & 0 & 0 & 6E5 & 0 & 0 \\ 0 & -7.11E4 & -1.07E7 & 0 & 7.11E4 & -1.07E7 \\ 0 & 1.07E7 & 1.07E9 & 0 & -1.07E7 & 2.13E9 \end{array}\right)
$$

$$
\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{c} u_2 \\ v_2 \\ \theta_2 \\ \hline u_3 \\ v_3 \\ \theta_3 \end{array}\right)
=
\left(\begin{array}{ccc|ccc} 6E5 & 0 & 0 & -6E5 & 0 & 0 \\ 0 & 7.11E4 & 1.07E7 & 0 & -7.11E4 & 1.07E7 \\ 0 & 1.07E7 & 2.13E9 & 0 & -1.07E7 & 1.07E9 \\ \hline -6E5 & 0 & 0 & 6E5 & 0 & 0 \\ 0 & -7.11E4 & -1.07E7 & 0 & 7.11E4 & -1.07E7 \\ 0 & 1.07E7 & 1.07E9 & 0 & -1.07E7 & 2.13E9 \end{array}\right)
\left(\begin{array}{c} u_2 \\ v_2 \\ \theta_2 \\ \hline u_3 \\ v_3 \\ \theta_3 \end{array}\right)
\tag{1E2.2}
$$

Bar 3

$$
\left(\begin{array}{c} S^3_{3x} \\ S^3_{3y} \\ M^3_3 \\ \hline S^3_{4x} \\ S^3_{4y} \\ M^3_4 \end{array}\right)
=
\left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{ccc|ccc} 9E5 & 0 & 0 & -9E5 & 0 & 0 \\ 0 & 2.4E5 & 2.4E7 & 0 & -2.4E5 & 2.4E7 \\ 0 & 2.4E7 & 3.2E9 & 0 & -2.4E7 & 1.6E9 \\ \hline -9E5 & 0 & 0 & 9E5 & 0 & 0 \\ 0 & -2.4E5 & -2.4E7 & 0 & 2.4E5 & -2.4E7 \\ 0 & 2.4E7 & 1.6E9 & 0 & -2.4E7 & 3.2E9 \end{array}\right)
$$

$$
\left(\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)
\left(\begin{array}{c} u_3 \\ v_3 \\ \theta_3 \\ \hline u_4 \\ v_4 \\ \theta_4 \end{array}\right)
=
\left(\begin{array}{ccc|ccc} 2.4E5 & 0 & 2.4E7 & -2.4E5 & 0 & 2.4E7 \\ 0 & 9E5 & 0 & 0 & -9E5 & 0 \\ 2.4E7 & 0 & 3.2E9 & -2.4E7 & 0 & 1.6E9 \\ \hline -2.4E5 & 0 & -2.4E7 & 2.4E5 & 0 & -2.4E7 \\ 0 & -9E5 & 0 & 0 & 9E5 & 0 \\ 2.4E7 & 0 & 1.6E9 & -2.4E7 & 0 & 3.2E9 \end{array}\right)
\left(\begin{array}{c} u_3 \\ v_3 \\ \theta_3 \\ \hline u_4 \\ v_4 \\ \theta_4 \end{array}\right)
\tag{1E2.3}
$$

As in the case of the plane truss, the first equations to consider are those that define the boundary conditions:

$$u_1 = 0; \quad v_1 = 0; \quad \theta_1 = 0$$

$$u_4 = 0; \quad v_4 = 0; \quad \theta_4 = 0$$

These boundary condition relationships can be used to establish a set of equations that represent the structural equilibrium of the nodes. This allows the removal of equations where reactions do not appear. Then it can be written as:

$$Node \quad 2: \quad \left\{ \begin{array}{c} F_{2x} \\ F_{2y} \\ M_2 \end{array} \right\} = \left\{ \begin{array}{c} S^1_{2x} + S^2_{2x} \\ S^1_{2y} + S^2_{2y} \\ M^1_2 + M^2_2 \end{array} \right\}$$

$$Node \quad 3: \quad \left\{ \begin{array}{c} F_{3x} \\ F_{3y} \\ M_3 \end{array} \right\} = \left\{ \begin{array}{c} S^2_{3x} + S^3_{3x} \\ S^2_{3y} + S^3_{3y} \\ M^2_3 + M^3_3 \end{array} \right\}$$

Thus the system of equilibrium equations (analogous to (1.6)) for this case is

$$\begin{pmatrix} 10000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} S^1_{2x} + S^2_{2x} \\ S^1_{2y} + S^2_{2y} \\ M^1_2 + M^2_2 \\ S^2_{3x} + S^3_{3x} \\ S^2_{3y} + S^3_{3y} \\ M^2_3 + M^3_3 \end{pmatrix} \tag{1E2.4}$$

If the expressions obtained for the reaction of the bars are substituted in (1E2.4), S^k_i, on the nodes as a function of internal displacement given by (1E2.1), (1E2.2) and (1E2.3), the following system of equations is obtained:

$$\begin{pmatrix} 10000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 8.4E5 & 0 & 2.4E7 & -6E5 & 0 & 0 \\ 0 & 9.711E5 & 1.07E7 & 0 & -7.11E4 & 1.07E7 \\ 2.4E7 & 1.07E7 & 5.33E9 & 0 & -1.07E7 & 1.07E9 \\ -6E5 & 0 & 0 & 8.4E5 & 0 & 2.4E7 \\ 0 & -7.11E4 & -1.07E7 & 0 & 9.711E5 & -1.07E7 \\ 0 & 1.07E7 & 1.07E9 & 2.4E7 & -1.07E7 & 5.33E9 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \\ \theta_2 \\ u_3 \\ v_3 \\ \theta_3 \end{pmatrix} \tag{1E2.5}$$

and the solution is

$$u_2 = 0.039 \quad cm; v_2 = 0.0029 \quad cm; \theta_2 = -1.61E-4 \quad rad$$