

# THE NATURAL REARRANGEMENT INVARIANT STRUCTURE ON TENSOR PRODUCTS

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ABSTRACT. We prove that the only rearrangement invariant (r.i.) spaces for which there exists a crossnorm verifying that the tensor product of these spaces preserves the “natural” r.i. space structure, in the sense that it makes the multiplication operator  $B$  a topological isomorphism, are the  $L_p$  spaces.

## 1. INTRODUCTION

The connections between tensor products and rearrangement invariant Banach spaces have been studied in several works (see for instance [11], [15] and the references therein). The general framework has been the next: Let  $X(\Omega_1)$ ,  $Y(\Omega_2)$  and  $Z(\Omega_1 \times \Omega_2)$  be Banach function spaces. In which cases is the bilinear operator

$$B : X(\Omega_1) \times Y(\Omega_2) \longrightarrow Z(\Omega_1 \times \Omega_2),$$

defined as  $B(x, y)(s, t) = x \otimes y(s, t) = x(s)y(t)$  for every  $(s, t) \in \Omega_1 \times \Omega_2$ , bounded? Or, equivalently, when do we have a continuous embedding  $X(\Omega_1) \hat{\otimes}_\pi Y(\Omega_2) \subseteq Z(\Omega_1 \times \Omega_2)$ ? Here  $\pi$  denotes the projective tensor norm on  $X(\Omega_1) \otimes Y(\Omega_2)$  (See Sec. 2 for definitions) and  $X(\Omega_1) \hat{\otimes}_\pi Y(\Omega_2)$  the completion of  $(X(\Omega_1) \otimes Y(\Omega_2), \|\cdot\|_\pi)$ .

The close connection between the continuity of the operator  $B$  and the stability problem of the integral operator (see [11], Part III) has motivated a deep research on this problem. Most of these works (see for instance [2], [12], [13], [14], [15], [17]) have focused their results on concrete and important spaces: Lorentz spaces, Orlicz spaces, Marcinkiewicz spaces and so on (see [10] for definitions).

More recently, in [3], this problem was studied on the general context of symmetric function spaces on  $[0,1]$ . The tensor product was there used to study the multiplier space  $\mathcal{M}(X)$  of  $X$ . In [4] the authors focused their work on rearrangement invariant (r.i.) Banach spaces on  $[0,1]$ , relating the

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multiplicator space  $\mathcal{M}(X)$  with properties of subspaces of  $X$  and, again, with the continuity of the operator  $B$  (see theorem 7).

In this work we change the point of view of the problem, and we study the natural question of which r.i. spaces  $X(\Omega_1), Y(\Omega_2)$  and which crossnorms  $\alpha$  verify that the tensor product  $X(\Omega_1) \hat{\otimes}_\alpha Y(\Omega_2)$  is again an r.i. space. As far as we know, there is not much work done about this problem, though some partial results were previously stated in [16]. A priori, there can be “many” r.i. structures on the same space  $X(\Omega_1) \hat{\otimes}_\alpha Y(\Omega_2)$ . Anyway, there exists a “natural” structure, associated to the *product operator*  $B$ , on  $\Omega_1 \times \Omega_2$ . We are asking then about the cases in which  $X(\Omega_1) \hat{\otimes}_\alpha Y(\Omega_2) = Z(\Omega_1 \times \Omega_2)$ , for some r.i. space  $Z$ .

Thus, we are now interested in the cases in which the operator  $B$  is not only continuous, but also a topological isomorphism. Specifically, the framework of this paper is the question:

*Which r.i. spaces  $X, Y, Z$  verify that there exists a crossnorm  $\alpha$  such that the operator  $\hat{B} : X(\Omega_1) \hat{\otimes}_\alpha Y(\Omega_2) \rightarrow Z(\Omega_1 \times \Omega_2)$  is a topological isomorphism from  $X(\Omega_1) \hat{\otimes}_\alpha Y(\Omega_2)$  onto  $Z(\Omega_1 \times \Omega_2)$ ? (where  $\hat{B}$  is the extension of the operator  $B$  to the completion  $X(\Omega_1) \hat{\otimes}_\alpha Y(\Omega_2)$ ). We will show that the only case is  $X = Y = Z = L_p$  and  $\alpha = \Delta_p$  for some  $p \in [1, \infty)$  (See Sec. 2 for definitions).*

Finally we want to mention that there is another line of study relating tensor products and ordered structures. In [7] and [8] the author started a research about the possibility of defining crossnorms on tensor products of Banach lattices in such a way that they preserve the lattice structure (see also [6]). A lot of work has been done in this direction, but the symmetric structure inherent to r.i. spaces makes our problem much more restrictive.

## 2. DEFINITIONS AND NOTATION

In this work we relate both theories: rearrangement invariant Banach spaces and tensor product crossnorms. We start with some definitions we will need. We refer to [10] for a complete work on r.i. theory and to [5] for the part of crossnorms.

Let  $(\Omega, \Sigma, \lambda)$  be a measure space. We also consider the product measure space denoted by  $(\Omega \times \Omega, \lambda \otimes \lambda)$ .

We denote by  $\mathcal{M}_0(\Omega)$  (resp.  $\mathcal{M}_0(\Omega \times \Omega)$ ) the set of measurable functions on  $\Omega$  (resp. on  $\Omega \times \Omega$ ) over  $\mathbb{K} = (\mathbb{R} \text{ or } \mathbb{C})$ .

Given a function  $f \in \mathcal{M}_0(\Omega)$ , we denote  $\mu_f$  the distribution function of  $f$ , defined by

$$\mu_f(x) := \lambda\{t \in \Omega : |f(t)| > x\},$$

for every  $x \geq 0$ . We will say that two measurable functions  $f, g$  on  $\Omega$  are equimeasurable if they have the same distribution function.

For a given  $f \in \mathcal{M}_0(\Omega)$ , the decreasing rearrangement of  $f$  is a function  $f^*$  defined on  $[0, \infty)$  by

$$f^*(t) := \inf\{x : \mu_f(x) \leq t\},$$

$t \in [0, \infty)$ .

A Banach function space  $X$  on  $\Omega$  is said to be a rearrangement invariant (r.i.) space if the next property holds:

*If we have  $f, g \in \mathcal{M}_0(\Omega)$  such that  $f^*(t) \leq g^*(t)$  for every  $t \in [0, \infty)$  and  $g \in X(\Omega)$ , then  $f \in X(\Omega)$  and  $\|f\|_X \leq \|g\|_X$ .*

It follows trivially that if  $X(\Omega)$  is an r.i. space on  $\Omega$  and  $f, g \in \mathcal{M}_0(\Omega)$  are equimeasurable, then  $f \in X(\Omega) \Leftrightarrow g \in X(\Omega)$  and, in this case,  $\|f\|_X = \|g\|_X$ .

Following ([10], 2.a) we will consider  $(\Omega, \Sigma, \lambda)$  a separable measure space. This implies that the study of the r.i. spaces over  $(\Omega, \Sigma, \lambda)$  reduces immediately to the cases  $\Omega = I = [0, 1]$  with the usual Lebesgue measure  $\lambda$ ,  $\Omega = [0, \infty)$  with the usual Lebesgue measure  $\lambda$ , and the case in which  $\Omega$  is the set of integers with the discrete measure. Actually, we will study this last case just for Banach spaces with symmetric basis. Then, in the first part of the work,  $\Omega$  will denote one of the first two cases.

When we have an r.i. space  $X = X(\Omega)$  on  $\Omega$ , the corresponding r.i. space  $X(\Omega \times \Omega)$  on  $\Omega \times \Omega$  is the space of measurable functions  $x(s, t)$  on  $\Omega \times \Omega$  such that  $x^*(t) \in X(\Omega)$ , with the norm  $\|x\|_{X(\Omega \times \Omega)} = \|x^*\|_{X(\Omega)}$ , where  $x^*$  denotes the decreasing rearrangement of  $x$ .

To agree with the results of [9] and [4], which we will use, we assume that every r.i.  $X$  is either separable or it has the Fatou property.

We will need to use that for every r.i. space  $X(I)$  we have continuous embeddings

$$L_\infty(I) \hookrightarrow X(I) \hookrightarrow L_1(I),$$

both with norm one. And for every r.i. space  $Y(0, \infty)$  we have continuous embeddings

$$L_\infty(0, \infty) \cap L_1(0, \infty) \hookrightarrow Y(I) \hookrightarrow L_\infty(0, \infty) + L_1(0, \infty)$$

also with norm one, and where the norms considered are  $\max(\|f\|_1, \|f\|_\infty)$  and  $\int_0^1 f^*(t)dt$  in the first and third spaces respectively (see [10] 2.a). It follows that we have  $\mathcal{S} \subset X(\Omega)$ , where  $\mathcal{S}$  denotes the set of simple functions on  $\Omega$ .

We will also need a few notions about tensor products and crossnorms. Given two Banach spaces  $X, Y$ , we say that  $\alpha$  is a reasonable crossnorm whenever it satisfies the conditions:

- (1)  $\alpha(x \otimes y) \leq \|x\| \|y\|$  for all  $x \in X$  and  $y \in Y$ , and
- (2) if  $x^* \in X^*$  and  $y^* \in Y^*$ , then  $x^* \otimes y^* \in (X \otimes Y, \alpha)^*$  and has functional norm  $\leq \|x^*\| \|y^*\|$ .

We will call just crossnorm to any reasonable crossnorm. It is very easy to check that, actually, both inequalities are equalities.

There are two particularly interesting crossnorms on the tensor product  $X \otimes Y$ . The projective crossnorm, defined by

$$\pi(u) = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| \right\},$$

where the infimum is taken over all the representations of  $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ . And the injective crossnorm, defined by

$$\varepsilon(u) = \sup \{ |\langle u, x^* \otimes y^* \rangle| : x^* \in B_{X^*}, y^* \in B_{Y^*} \}.$$

Here  $B_{X^*}$  denotes the closed unit ball of the dual space  $X^*$  of  $X$  (and the same for  $Y$ ). It follows easily that every crossnorm  $\alpha$  satisfies  $\varepsilon \leq \alpha \leq \pi$ .

We need to define a family of crossnorms which will be crucial in our results. Let  $(\Omega, \mu)$  be an arbitrary measure space and  $E$  a normed space. Then, following ([5], Sec. 7), for any  $p \in [1, \infty)$  we consider the spaces of (classes of a.e. equal) Bochner  $p$ -Integrable functions  $\Omega \rightarrow \hat{E}$ ,  $L_p(\mu, \hat{E})$ . If we consider the injective natural mapping

$$L_p(\mu) \otimes E \hookrightarrow L_p(\mu, \hat{E})$$

defined by  $\tilde{f} \otimes x \mapsto \tilde{f}(\cdot)x$ , we can define the crossnorm

$$\Delta_p(f; L_p, E) := \left( \int_{\Omega} \|f(w)\|_E^p d\mu(w) \right)^{\frac{1}{p}}$$

on  $L_p \otimes E$ . We denote it by  $L_p \otimes_{\Delta_p} E$  and by  $L_p \hat{\otimes}_{\Delta_p} E$  its completion. It is not difficult to see that  $\Delta_1 = \pi$  on  $L_1 \otimes E$  and, using a density argument with the simple functions, it follows that  $L_p \hat{\otimes}_{\Delta_p} E = L_p(\mu, \hat{E})$  is isometrically isomorphic. In particular, given two arbitrary measure spaces  $(\Omega_1, \mu_1)$  and  $(\Omega_2, \mu_2)$ , for every  $1 \leq p < \infty$  we have the isometric identifications

$$L_p(\mu_1 \otimes \mu_2) = L_p(\mu_1) \hat{\otimes}_{\Delta_p} L_p(\mu_2) = L_p(\mu_1, L_p(\mu_2)).$$

Finally, in the case  $p = \infty$ ,  $L_{\infty}(\mu, \hat{E})$  is the space of (classes of locally a.e. equal) bounded  $\mu$ -measurable functions  $\Omega \rightarrow \hat{E}$ . With the same natural mapping as above we define now

$$\Delta_{\infty}(f; L_{\infty}, E) := \text{ess-sup} \|f(\cdot)\|_E.$$

It is easy to see that  $\Delta_{\infty} = \varepsilon$  on  $L_{\infty} \otimes E$ .

**Remark 2.1.** We have to notice here that the isometric identification above is not onto in the case  $p = \infty$ . It is well known that  $L_{\infty}(\Omega) \hat{\otimes}_{\varepsilon} L_{\infty}(\Omega) \subsetneq L_{\infty}(\Omega \times \Omega)$  in the cases we are considering.

## 3. RESULTS

As we mentioned before we want to characterize which r.i. Banach spaces  $X$ ,  $Y$  and  $Z$  verify that there exists a crossnorm  $\alpha$  such that the operator

$$B : X(\Omega) \otimes_{\alpha} Y(\Omega) \longrightarrow Z(\Omega \times \Omega),$$

defined as  $B(x \otimes y)(s, t) = x(s)y(t)$  for every  $(s, t) \in \Omega \times \Omega$ , is an onto topological isomorphism when we extend it to the completion  $X(\Omega) \hat{\otimes}_{\alpha} Y(\Omega)$ .

By the definition of the crossnorms  $\Delta_p$ , it is obvious that when we consider  $X = Y = L_p$  and  $\alpha = \Delta_p$  for  $1 \leq p < \infty$ , the statement holds. The main theorem of this work states that the converse is also true; that is

**Theorem 3.1.** *Given  $X$ ,  $Y$ ,  $Z$  r.i. spaces. Then the operator  $\hat{B}$  is a topological isomorphism from  $X(\Omega) \hat{\otimes}_{\alpha} Y(\Omega)$  onto  $Z(\Omega \times \Omega)$  if and only if there exists a  $p \in [1, \infty)$  such that  $X = Y = Z = L_p$  and  $\alpha = \Delta_p$ .*

We will first treat the continuous cases of  $\Omega$  ( $\Omega = I = [0, 1]$  and  $\Omega = [0, \infty)$ ). These two cases admit almost the same proof, so we will show the first one, and we will indicate the slight modifications required in the case  $[0, \infty)$ .

Before proving the result we need the following lemma:

**Lemma 3.2.** *There is not any r.i. space  $Z$  and crossnorm  $\alpha$  such that the operator  $\hat{B}$  is a topological isomorphism from  $L_{\infty}(I) \hat{\otimes}_{\alpha} L_{\infty}(I)$  onto  $Z(I \times I)$ .*

*Proof.* Using that  $\hat{B} : L_{\infty}(I) \hat{\otimes}_{\alpha} L_{\infty}(I) \longrightarrow Z(I \times I)$  is an onto topological isomorphism and  $Z$  is an r.i. space, we know that the inclusion  $id : L_{\infty}(I) \hookrightarrow Z(I)$  is a topological isomorphism. And, thus,  $id : L_{\infty}(I \times I) \hookrightarrow Z(I \times I)$  is too.

Given an element  $a \in L_{\infty}(I) \otimes L_{\infty}(I)$ , we have

$$\|a\|_{L_{\infty}(I) \otimes_{\alpha} L_{\infty}(I)} \sim \|B(a)\|_{Z(I \times I)} \sim \|B(a)\|_{L_{\infty}(I \times I)} = \|a\|_{L_{\infty}(I) \otimes_{\varepsilon} L_{\infty}(I)},$$

where  $\sim$  denotes equivalence between the norms.

Since  $\varepsilon < \alpha$  always holds, this says that  $\alpha \sim \varepsilon$  on  $L_{\infty}(I) \otimes L_{\infty}(I)$ , and thus the completion is the same for both crossnorms. The remark 2.1 completes the proof.  $\square$

**Remark 3.3.** It is trivial that the proof above does not depend on the cases of  $\Omega$  that we are considering.

Following [4], given an r.i. space  $X$  on  $I$ , we denote

$$V_0(X) = \{a \in X : a \neq 0, a = a^*\}.$$

Now, for any function  $a \in V_0(X)$  and dyadic intervals  $\Delta_{n,k} = [\frac{k-1}{2^n}, \frac{k}{2^n}]$ ,  $k = 1, 2, \dots, 2^n$ ,  $n \in \mathbb{N}$ , we consider the dilations and translations of the function  $a$ :

$$a_{n,k} = \begin{cases} a(2^n t - k + 1) & \text{if } t \in [\frac{k-1}{2^n}, \frac{k}{2^n}], \\ 0 & \text{otherwise.} \end{cases}$$

It follows then that the support of  $a_{n,k}$  is contained in  $\Delta_{n,k}$ , and for every  $x > 0$  we have

$$(1) \quad \lambda(\{t \in \Delta_{n,k} : |a_{n,k}(t)| > x\}) = \frac{1}{2^n} \lambda(\{t \in I : |a(t)| > x\}).$$

The key of the proof of our theorem 3.1 is the next result from [4].

**Theorem 3.4.** [4] *(Theorem 7) Let  $X$  be an r.i. space on  $[0, 1]$ . Then, there exists a  $p \in [1, \infty]$  such that  $X = L_p$  if and only if there exists a constant  $C > 0$  such that*

$$(2) \quad C^{-1} \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \|X\| a \|X\right\| \leq \left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \|X\right\| \leq C \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \|X\| a \|X\right\|,$$

for all  $a \in V_0(X)$  and all  $c_{n,k} \in \mathbb{R}$  with  $k = 1, 2, \dots, 2^n, n = 0, 1, 2, \dots$ .

With this at hand we can prove the next proposition:

**Proposition 3.5.** *Given  $X, Y, Z$  r.i. spaces and  $\alpha$  a crossnorm, if the operator  $B$  is a topological isomorphism from  $X(I) \otimes_\alpha Y(I)$  into  $Z(I \times I)$ , then there must exist a  $p \in [1, \infty]$  such that  $X = L_p = Y$ .*

*Proof.* Using that  $B$  is a topological isomorphism, we have that there exists a constant  $M$  such that for every  $x \in X(I) \otimes Y(I)$  it holds that

$$(3) \quad M^{-1} \alpha(x) \leq \|B(x)\|_{Z(I \times I)} \leq M \alpha(x).$$

Since  $Z$  is an r.i. space, it is trivial to see that the mapping  $f \rightarrow f \otimes \mathbb{1}$  from  $Z(I)$  into  $Z(I \times I)$  is a linear isometry (where  $\mathbb{1}$  denotes the characteristic function on  $I$ ). Using that  $B$  is a topological isomorphism, it follows that  $id : X(I) \hookrightarrow Z(I)$  is a topological isomorphism with the same constant  $M$  as  $B$  (and the same holds for  $Y$ ). We want to remark the next trivial fact, that we will use later:

If we consider the set of simple functions  $\mathcal{S}$  on  $I$ , we have that  $\mathcal{S} \subset X(I) \cap Y(I) \subset Z(I)$  and, by the comments above, the norms  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  are equivalent on  $\mathcal{S}$  with constant  $M^2$  (in particular, for every  $s \in \mathcal{S}$ ,  $\|s\|_Y \leq M^2 \|s\|_X$ ).

Suppose there is no  $p \in [1, \infty]$  such that  $X = L_p$ ; we want to reach a contradiction. At least one of the inequalities in (2) must fail. We assume the inequality on the right fails (the reasoning in the other case is analogous). Then, there exists a function  $a \in V_0(X)$ , a natural number  $n \in \mathbb{N}$  and some coefficients  $c_{n,k} \in \mathbb{R}$  with  $k = 1, 2, \dots, 2^n$ , such that

$$\left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \|X\right\| > M^4 \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \|X\| a \|X\right\|.$$

We consider the next two elementary tensors in  $X(I) \otimes Y(I)$ :

$$x = \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \otimes \mathbb{1}, \text{ and } y = a \otimes \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}}.$$

We are going to show that  $\mu_f = \mu_g$  on  $Z(I \times I)$ , where  $f = B(x)$  and  $g = B(y)$ . Then, using that  $Z$  is an r.i. space, we will have that  $\|B(x)\|_{Z(I \times I)} = \|B(y)\|_{Z(I \times I)}$ , and thus  $\alpha(x) \leq M^2 \alpha(y)$ . This will be a contradiction because we have chosen the elements  $x$  and  $y$  such that

$$\begin{aligned} \alpha(x) &= \left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X \|\mathbb{1}\|_Y = \left\| \sum_{k=1}^{2^n} c_{n,k} a_{n,k} \right\|_X > \\ &> M^4 \|a\|_X \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_X \geq M^2 \|a\|_X \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_{\Delta_{n,k}} \right\|_Y = M^2 \alpha(y). \end{aligned}$$

where in the last inequality we have used the inequality described before for simple functions on  $X \cap Y$ .

Let us prove it. Let  $w > 0$ . On one hand

$$\begin{aligned} \mu_f(w) &= \lambda(\{(s, t) \in I \times I : \left| \sum_{k=1}^{2^n} c_{n,k} a_{n,k}(s) \mathbb{1}(t) \right| > w\}) \\ &= \lambda(\{s \in I : \left| \sum_{k=1}^{2^n} c_{n,k} a_{n,k}(s) \right| > w\}) = \sum_{k=1}^{2^n} \lambda\{s \in \Delta_{n,k} : |a_{n,k}(s)| > \frac{w}{|c_{n,k}|}\} \\ &= \frac{1}{2^n} \sum_{k=1}^{2^n} \lambda\{s \in I : |a(s)| > \frac{w}{|c_{n,k}|}\}. \end{aligned}$$

We have used (1) in the last step.

On the other hand,

$$\begin{aligned} \mu_g(w) &= \lambda(\{(s, t) \in I \times I : \left| \sum_{k=1}^{2^n} c_{n,k} a(s) \chi_{\Delta_{n,k}}(t) \right| > w\}) \\ &= \sum_{k=1}^{2^n} \lambda\{(s, t) \in \Delta_{n,k} \times I : |c_{n,k} a(s)| > w\} \\ &= \sum_{k=1}^{2^n} \lambda(\Delta_{n,k}) \lambda\{s \in I : |a(s)| > \frac{w}{|c_{n,k}|}\} = \frac{1}{2^n} \sum_{k=1}^{2^n} \lambda\{s \in I : |a(s)| > \frac{w}{|c_{n,k}|}\}. \end{aligned}$$

Hence there must be a  $p \in [1, \infty]$  such that  $X = L_p$ . Similarly we can proceed for  $Y$ , and get that there must be a  $q \in [1, \infty]$  with  $Y = L_q$ . And, since  $\mathcal{S}$  is dense in both  $X(I)$  and  $Y(I)$  and the norms are equivalent on the elements of  $\mathcal{S}$ , we can conclude that  $p = q$  and  $X = Y$ .  $\square$

With that and some density arguments we can prove the main result:

*Proof.* of theorem 3.1

One of the implications is trivial, let us proceed with the other one.

From the previous theorem we already know that necessarily there exists a  $p \in [1, \infty]$  such that  $X = L_p(I) = Y$ . Now, by lemma 3.2 we can rule out the case  $X = Y = L_\infty(I)$ .

The set  $S$  of simple functions on  $I$  is dense in  $L_p(I)$ , and for every cross-norm  $\alpha$ , the set  $S \otimes S$  is dense in  $L_p(I) \otimes_\alpha L_p(I)$ , and hence so is  $B(S \otimes S)$  in  $Z(I \times I)$ . Then, we have  $B(S \otimes S) = S \otimes S$  dense in  $L_p(I) \hat{\otimes}_{\Delta_p} L_p(I) = L_p(I \times I)$ , and this space isomorphically embedded (by the identity) into  $Z(I \times I)$  and, hence, dense in  $Z(I \times I)$ . Therefore  $Z$  must be also  $L_p$  and the norm  $\alpha$  must be equivalent to the norm  $\Delta_p$  just by the definition of this crossnorm (see Sec. 2). □

The proof of the case  $\Omega = [0, \infty)$  can be done following the same steps using (Theorem 5.4, [9]). We have to mention that in this theorem they need to add the hypothesis  $\phi_E(0+) = \phi_{E'}(0+) = 0$ .

Anyway, we only need to use the equivalence between (ii) and (iv), and this is also true without this hypothesis. We will explain this a bit later.

We follow the same notation as in [9]. Given an r.i. space  $X$  on  $[0, \infty)$ , we denote

$$V(X) = \{a \in X : a \neq 0, \text{supp } a \subset [0, 1), a = a^*\}.$$

Then, fixed an element  $a \in V(X)$ , we consider the translation of  $a(t)$  to the interval  $[k-1, k)$  for every  $k \geq 1$ , i.e.

$$a_k(t) = \begin{cases} a(t - (k-1)) & \text{if } t \in [k-1, k), \\ 0 & \text{otherwise.} \end{cases} \quad \text{for every } k \geq 1.$$

Then, we have

**Theorem 3.6.** [9] (Theorem 5.4) *Let  $X$  be an r.i. space on  $[0, \infty)$ . Then, there exists a  $p \in [1, \infty]$  such that  $X = L_p$  if and only if there exists a constant  $C > 0$  such that*

$$(4) \quad C^{-1} \left\| \sum_{k=1}^n c_k \chi_{[k-1, k)} \right\|_X \|a\|_X \leq \left\| \sum_{k=1}^n c_k a_k \right\|_X \leq C \left\| \sum_{k=1}^n c_k \chi_{[k-1, k)} \right\|_X \|a\|_X,$$

for every natural  $n \in \mathbb{N}$ , every  $a \in V(X)$  and all  $c_k$  with  $k = 1, 2, \dots$ .

*Proof.* We only prove the left to right implication. The other is easy.

Following exactly the same way as in [9] (Theorem 5.4), from (4) it follows that the fundamental function of  $X$  verifies  $\phi_X(t) \approx t^\alpha$  for some  $\alpha \in [0, 1]$  (we do not rule out the case  $\alpha = 0$  in [9] (Theorem 5.2)). If we are in the case  $\alpha \in (0, 1)$ , we continue the proof as in [9] and we get  $X = L_p$  for some  $p \in (1, \infty)$ . For  $\alpha = 0, 1$  it is known that the only spaces with these fundamental functions are  $L_1$  and  $L_\infty$ . Note that the space  $\Gamma = \overline{S}^{\|\cdot\|_\infty}$ , which



is the only space besides  $L_\infty$  which corresponds to  $\alpha = 0$ , is not considered since it is not separable and doesn't have the Fatou property.  $\square$

The case  $[0, \infty)$  follows now with exactly the same proof than in the case  $I$ .

*Symmetric basis on Banach spaces.* In the case we consider  $\Omega$  as the set of natural numbers with the discrete measure, the problem on separable spaces can be translated into bases. The question would then be: which Banach spaces  $X$  and  $Y$  with symmetric bases and which crossnorms  $\alpha$  can be put together so that the product basis is a symmetric basis of the tensor product  $X \hat{\otimes}_\alpha Y$ ? Some steps in the proof are now much easier because we have the notion of basis on  $X \hat{\otimes}_\alpha Y$ , while in the r.i. spaces we have to embed the tensor product into a space of functions on  $\Omega \times \Omega$  to define the r.i. structure.

We recall that a basis  $\{x_n\}_{n=1}^\infty$  of a Banach space  $X$  is said to be symmetric if for every permutation  $\pi$  of the integers  $\{x_{\pi(n)}\}_{n=1}^\infty$  is equivalent to  $\{x_n\}_{n=1}^\infty$ . A positive constant  $K$ , the symmetric basis constant of  $B$ , can be found as the supremum of the norms of these equivalences. The space can be given an equivalent norm so that the symmetric constant of this basis turns to be one, which leads immediately to an r.i. structure on the space.

Let  $X$  and  $Y$  be Banach spaces with symmetric bases  $\{x_n\}_{n=1}^\infty$  and  $\{y_n\}_{n=1}^\infty$  respectively, and let  $\alpha$  be a crossnorm. Then the product basis in  $Z = X \hat{\otimes}_\alpha Y$  is  $\{x_n \otimes y_m\}_{n,m=1}^\infty$ , together with certain order (see [10]) which makes  $\{x_n \otimes y_1\}_{n=1}^\infty$  and  $\{x_1 \otimes y_m\}_{m=1}^\infty$  subsequences of the product basis sequence.

**Remark 3.7.** When having a symmetric basis all its subsequences are equivalent to the original sequence. Hence, if the product basis is a symmetric basis of the tensor product then  $\{x_n\}_{n=1}^\infty$ ,  $\{y_n\}_{n=1}^\infty$  and  $\{x_n \otimes y_m\}_{n,m=1}^\infty$  are equivalent, and consequently  $X$ ,  $Y$  and  $Z$  are the same space.

The main theorem for the symmetric basis case can be proven with the same ideas before. The characterization of the spaces  $\ell_p$  needed now can be found in [1]: there must exist a constant  $K > 0$  such that

$$K^{-1} \|a\| \cdot \|b\| \leq \left\| \sum_{i,j=1}^{\infty} a_i b_j e_{i,j} \right\| \leq K \|a\| \cdot \|b\|$$

for all vectors  $a = \sum_{i=1}^{\infty} a_i e_i$ ,  $b = \sum_{j=1}^{\infty} b_j e_j$  in the space, and  $\{e_{i,j}\}_{j=1}^\infty$ ,  $i \in \mathbb{N}$  disjoint subsequences of the basis. The characterizations we have used in the cases before are, in fact, generalizations of this one.

**Theorem 3.8.** *If  $X$ ,  $Y$  are spaces with a symmetric basis  $\{x_n\}$  and  $\{y_n\}$  respectively, such that there exists a crossnorm  $\alpha$  that makes  $\{x_n \otimes y_m\}$  a symmetric basis in  $Z = X \hat{\otimes}_\alpha Y$ , then  $X = Y = Z$  is the space  $\ell_p$  and  $\alpha = \Delta_p$  for some  $1 \leq p < \infty$ , or it is the space  $c_0$  and  $\alpha = \varepsilon$ .*

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