A GENERALIZATION OF THE MONOTONE METHOD FOR SECOND ORDER PERIODIC BOUNDARY VALUE PROBLEM WITH IMPULSES AT FIXED POINTS

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Abstract. We present an existence result and some comparison results for a second order boundary value problem for ordinary differential equations with impulses at fixed moments. Then, it is presented the monotone iterative scheme. Finally we study the validity of the monotone iterative technique for a discontinuous nonlinearity in the dependent variable.

AMS (MOS) subject classification: 34A37, 34C25.

\(^*\)This is the preprint version of the paper published in Dynamics of continuous, discrete and impulsive systems, Vol. 7 (2000) 145-158.
1 Introduction

We consider the following general second order boundary value problem with impulses at fixed points:

\[
\begin{align*}
    u''(t) &= f(t, u(t)), & \text{a.e. } t \in J' = J \setminus \{t_1, t_2, \ldots, t_p\} \\
    \Delta u'(t_k) &= L_k(u(t_k), u'(t_k)), & k = 1, 2, \ldots, p \\
    \Delta u(t_k) &= \tilde{L}_k(u(t_k), u'(t_k)), & k = 1, 2, \ldots, p \\
    u(0) - u(T) &= \lambda_0, & u'(0) - u'(T) = \lambda_1,
\end{align*}
\]

where \( J = [0, T], 0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T, \) \( f: J \times \mathbb{R} \to \mathbb{R}, \)
\( L_k, \tilde{L}_k: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \) \( \Delta u'(t_k) = u'(t_k^+) - u'(t_k^-), \) \( \Delta u(t_k) = u(t_k^+) - u(t_k^-), \)
\( k = 1, 2, \ldots, p, \lambda_1, \lambda_0 \in \mathbb{R}. \)

We shall prove an existence result for (1.1) by using the Banach contraction principle.

Note that when \( \lambda_0 = \lambda_1 = 0 \) we have a periodic boundary value problem.

In order to define the concept of solution for (1.1) we introduce the following sets of functions

\[
\begin{align*}
    PC(J) &= \{ u: J \to \mathbb{R} : u \in C(J'), \text{ there exist } u(0^+) = u(0), u(t_k^+) \text{ and } u(t_k^-) = u(t_k), k = 1, 2, \ldots, p + 1 \}. \\
    PC^1(J) &= \{ u \in PC(J) : u|_{(t_k, t_{k+1})} \in C^1(t_k, t_{k+1}), \text{ there exist } u'(0^+), u'(T^-), u'(t_k^+), u'(t_k^-), \text{ and } u'(t_k), k = 1, 2, \ldots, p \}. \\
    \Omega^{2,1}(J) &= \{ u \in PC^1(J) : u|(t_k, t_{k+1}) \in W^{2,1}(t_k, t_{k+1}), k = 0, 1, \ldots, p \}. 
\end{align*}
\]

Note that \( PC(J), PC^1(J) \) and \( \Omega^{2,1}(J) \) are Banach spaces with the norms

\[
\|u\|_{PC} = \sup \{ |u(t)| : t \in J \}, \quad \|u\|_{PC^1} = \max \{ \|u\|_{PC}, \|u'\|_{PC} \},
\]

and

\[
\|u\|_{\Omega^{2,1}} = \max_{0 \leq k \leq p} \{ \|u|(t_k, t_{k+1})\| W^{2,1}(t_k, t_{k+1}) \},
\]

respectively.

By a solution of (1.1) we mean a function \( u \in \Omega^{2,1}(J) \) satisfying (1.1).

We shall prove an existence result for (1.1) by using the Banach contraction principle.
It is well known the importance of the monotone iterative technique to show the existence of solution of a nonlinear problem and to approximate maximal and minimal solutions of it in certain sector [7, 10, 11, 13]. To prove this type of results it is necessary, in general, to use a maximum principle or comparison result coupled with an existence result to guarantee the existence of monotone sequences that converge to the maximal and minimal solutions.

However, it is very difficult to develop a monotone iterative technique for the general impulsive problem (1.1), and, as far as we know, there are not works addressing this general problem. But there are some papers studying particular cases of it. For example in [9, 15] it is considered the case when \( L_k(x, y) = I_k(y) - y \) and \( \tilde{L}_k(x, y) = \tilde{I}_k(x) - x \), in [3] it is considered the case when \( L_k(x, y) = I_k(x) - y \) and \( \tilde{L}_k(x, y) = \tilde{I}_k(x) \), and in [4, 17] the authors considered the case when \( L_k(x, y) = I_k(x) \) and \( \tilde{L}_k(x, y) \) is constant with \( I_k, \tilde{I}_k : \mathbb{R} \to \mathbb{R} \) continuous functions for each \( k = 1, 2, \ldots, p \).

In this paper we shall develop a monotone iterative technique for the latter situation, i.e., for the problem

\[
\begin{align*}
  u''(t) &= f(t, u(t)), & \text{a.e. } t \in J' \\
  \Delta u'(t_k) &= I_k(u(t_k)), & k = 1, 2, \ldots, p \\
  \Delta u(t_k) &= \mu_k, & k = 1, 2, \ldots, p \\
  u(0) - u(T) &= \lambda_0, & u'(0) - u'(T) = \lambda_1,
\end{align*}
\]

with \( \mu_k \in \mathbb{R} \), for each \( k = 1, 2, \ldots, p \).

We say that \( \alpha, \beta \in \Omega^{2,1}(J) \) are classical lower and upper solutions of (1.2) respectively if they satisfy

\[
\begin{align*}
  \alpha''(t) &\geq f(t, \alpha(t)), & \text{a.e. } t \in J' \\
  \Delta \alpha'(t_k) &\geq I_k(\alpha(t_k)), & k = 1, 2, \ldots, p \\
  \Delta \alpha(t_k) &= \mu_k, & k = 1, 2, \ldots, p \\
  \alpha(0) - \alpha(T) &= \lambda_0, & \alpha'(0) - \alpha'(T) \geq \lambda_1,
\end{align*}
\]

and

\[
\begin{align*}
  \beta''(t) &\leq f(t, \beta(t)), & \text{a.e. } t \in J' \\
  \Delta \beta'(t_k) &\leq I_k(\beta(t_k)), & k = 1, 2, \ldots, p \\
  \Delta \beta(t_k) &= \mu_k, & k = 1, 2, \ldots, p \\
  \beta(0) - \beta(T) &= \lambda_0, & \beta'(0) - \beta'(T) \leq \lambda_1.
\end{align*}
\]

In recent works, related to first order impulsive equations and to first and second order non-impulsive equations, several authors have considered more general concepts of lower and upper solutions [1, 2, 5, 12, 14].

The principal quality of the lower and upper solutions that we shall define is that inequalities in the latter definition will be improved.

This work improves and complements some results of [2, 4, 9, 15, 17].
2 Existence and Comparison Results.

Now, we consider the following problem,

\[ u''(t) - M^2 u(t) = \sigma(t), \quad \text{a.e. } t \in J' \]
\[ \Delta u'(t_k) = L_k(u(t_k), u'(t_k)), \quad k = 1, 2, \ldots, p \]
\[ \Delta u(t_k) = L_k(u(t_k), u'(t_k)), \quad k = 1, 2, \ldots, p \]
\[ u(0) - u(T) = \lambda_0, \quad u'(0) - u'(T) = \lambda_1 \]

(2.3)

where \( M \in \mathbb{R}, M > 0 \) and \( \sigma \in L^1(J) \).

We have the following useful result

**Theorem 2.1** \( u \in \Omega^{2,1}(J) \) is a solution of problem (2.3) if and only if \( u \in PC^1(J) \) and it is a solution of the following impulsive integral equation

\[
 u(t) = \int_0^T G(t, s) \sigma(s) ds + G(t, 0) \lambda_1 + H(t, 0) \lambda_0 + \sum_{k=1}^p \left[ G(t, t_k) L_k(u(t_k), u'(t_k)) + H(t, t_k) \tilde{L}_k(u(t_k), u'(t_k)) \right],
\]

(2.4)

where \( G \) and \( H \) are defined by

\[
 G(t, s) = -\frac{1}{2M(e^{MT} - 1)} \left\{ \frac{e^{M(T-t+s)} + e^{M(t-s)}}{e^{M(T+t-s)} + e^{M(s-t)}}, \quad 0 \leq s < t \leq T \right. \\
 H(t, s) = \frac{1}{2(e^{MT} - 1)} \left\{ \frac{e^{M(T-t+s)} - e^{M(t-s)}}{e^{M(s-t)} - e^{M(T+t-s)}}, \quad 0 \leq t \leq s \leq T \right. 
\]

(2.5)

**Proof:** The proof is essentially the same that appears in similar results in [4, 17]. The only difference is that we consider more general impulsive functions but this does not affect the proof. \( \square \)

We indicate some important properties of functions \( G \) and \( H \):

\[
 G = \min_{(t,s) \in J \times J} \{ G(t, s) \} = -\frac{(1 + e^{MT})}{2M(e^{MT} - 1)} < 0
\]
\[
 H = \max_{(t,s) \in J \times J} \{ G(t, s) \} = -\frac{e^{MT}}{M(e^{MT} - 1)} < 0
\]

(2.5)

\[
 \min_{(t,s) \in J \times J} H(t, s) = -\frac{1}{2}, \quad \max_{(t,s) \in J \times J} H(t, s) = \frac{1}{2}.
\]
Also it is valid the following identity for each \( t \in J \):
\[
\int_0^T G(t,s)ds = -\frac{1}{M^2}.
\]

Using the representation (2.4) and the Banach contraction principle, we prove an existence and uniqueness result for (1.1) when the nonlinearity \( f \) satisfies
\[
f(t,u) = M^2 u + g(t,u), \quad \text{a.e.} \quad t \in J, \quad u \in \mathbb{R}, \quad (2.6)
\]
with \( M > 0 \) and \( g: J \times \mathbb{R} \to \mathbb{R} \) a Carathéodory function verifying certain Lipschitz condition.

**Theorem 2.2** Suppose that (2.6) holds and that there exist \( c_k, \bar{c}_k, \bar{d}_k \in \mathbb{R}^+ \), \( k = 1, 2, \ldots, p \) such that the functions \( g, \tilde{L}_k \) and \( L_k, \bar{L}_k \) verify the following conditions
\[
|L_k(x_1, y_1) - L_k(x_2, y_2)| \leq c_k|x_1 - x_2| + d_k|y_1 - y_2|,
\]
\[
|\tilde{L}_k(x_1, y_1) - \tilde{L}_k(x_2, y_2)| \leq \bar{c}_k|x_1 - x_2| + \bar{d}_k|y_1 - y_2|,
\]
and
\[
|g(t, x_1) - g(t, x_2)| \leq c|x_1 - x_2|,
\]
for each \( x_1, x_2, y_1, y_2 \in \mathbb{R} \).

Let
\[
r_1 = \frac{c}{M^2} + \sum_{k=1}^p \left[ \frac{\bar{c}_k + \bar{d}_k}{2} - G(c_k + d_k) \right]
\]
and
\[
r_2 = \frac{Tc}{2} + \sum_{k=1}^p \left[ \frac{c_k + d_k}{2} - M^2 G(\bar{c}_k + \bar{d}_k) \right].
\]

If \( r = \max\{r_1, r_2\} < 1 \), then there exists a unique solution of the problem (1.1).

**Proof:** We define the operator \( S: PC^1(J) \to PC^1(J) \) by
\[
[Su](t) = \int_0^T G(t,s)g(s,u(s))ds + G(t,0)\lambda_1 + H(t,0)\lambda_0
\]
\[
+ \sum_{k=1}^p \left[ G(t,t_k)L_k(u(t_k), u'(t_k)) + H(t,t_k)\tilde{L}_k(u(t_k), u'(t_k)) \right]. \tag{2.7}
\]

It is easy to see that, in the hypotheses of the theorem, \( u \) is a solution of (2.3) if and only if \( u \) is a fixed point of \( S \).
Now, we show that $S$ is a contraction in $PC^1(J)$. Indeed, consider $u, v \in PC^1(J)$, thus

$$
\| (Su - Sv)(t) \| = \left| \int_0^T G(t, s)[g(s, u(s)) - g(s, v(s))]ds 
+ \sum_{k=1}^p \{ G(t, t_k)[L_k(u(t_k), u'(t_k)) - L_k(v(t_k), v'(t_k))] 
+ M^2 G(t, t_k)[\tilde{L}_k(u(t_k), u'(t_k)) - \tilde{L}_k(v(t_k), v'(t_k))] \right|
\leq \frac{c\|u - v\|_{PC}}{M^2} + \sum_{k=1}^p \left( c_k |u(t_k) - v(t_k)| + d_k |u'(t_k) - v'(t_k)| \right)
+ \left( \frac{c_k}{2} + \frac{d_k}{2} \right) \| u, v \|_{PC^1}, \quad t \in J.
$$

On the other hand

$$
\| (Su)' - (Sv)'(t) \| = \left| \int_0^T H(t, s)[g(s, u(s)) - g(s, v(s))]ds 
+ \sum_{k=1}^p \{ H(t, t_k)[L_k(u(t_k), u'(t_k)) - L_k(v(t_k), v'(t_k))] 
+ M^2 H(t, t_k)[\tilde{L}_k(u(t_k), u'(t_k)) - \tilde{L}_k(v(t_k), v'(t_k))] \right|
\leq \frac{T_c}{2} \| u - v \|_{PC} + \sum_{k=1}^p \left( c_k |u(t_k) - v(t_k)| + d_k |u'(t_k) - v'(t_k)| \right)
+ \left( \frac{c_k}{2} + \frac{d_k}{2} \right) \| u, v \|_{PC^1}, \quad t \in J.
$$

Therefore,

$$
\| Su - Sv \|_{PC^1} \leq \max\{r_1, r_2\} \| u - v \|_{PC^1} = r \| u - v \|_{PC^1},
$$

showing that $S$ is a contraction and therefore it has a unique fixed point. $\Box$

When the impulsive functions are constants and $g(t, u) = \sigma(t)$, a.e. $t \in J$, i.e.,

$$
\begin{align*}
    u''(t) - M^2 u(t) & = \sigma(t), \quad \text{a.e. } t \in J' \\
    \Delta u'(t_k) & = \mu_{1k}, \quad k = 1, 2, \ldots, p \\
    \Delta u(t_k) & = \mu_{0k}, \quad k = 1, 2, \ldots, p
\end{align*}
$$

(2.8)

with $\mu_{0k}, \mu_{1k} \in \mathbb{R}$, $k = 1, 2, \ldots, p$. As a consequence of the previous Theorem, we have the following result.
Corollary 2.1 Problem (2.8) has a unique solution $u \in \Omega^{2,1}(J)$, for any $\sigma \in L^1(J)$, $\lambda_0, \lambda_1 \in \mathbb{R}$ and $\mu_{0k}, \mu_{1k} \in \mathbb{R}$, $k = 1, 2, \ldots, p$.

Now we give a comparison result. We note that this result improves Lemma 3.6 in [2] in the periodic case without impulses.

For $\sigma \in L^1(J)$ we write $\sigma = \sigma^+ - \sigma^-$, $\sigma^+ = \max\{\sigma, 0\}$, $\sigma^- = -\min\{\sigma, 0\}$

Lemma 2.1 (i) Let $\sigma \in L^1(J)$ be such that

$$G \int_0^T \sigma^+(s) ds - \mathcal{G} \int_0^T \sigma^-(s) ds \leq \mathcal{G}(\lambda_1^- + \sum_{k=1}^p \mu_{-1k}^-) - \mathcal{G}(\lambda_1^+ + \sum_{k=1}^p \mu_{1k}^+) - \frac{1}{2}(|\lambda_0| + \sum_{k=1}^p |\mu_{0k}|)$$

then the unique solution of (2.8) satisfies $u \leq 0$.

(ii) Similarly, let $\sigma \in L^1(J)$ be such that

$$\mathcal{G} \int_0^T \sigma^+(s) ds - \mathcal{G} \int_0^T \sigma^-(s) ds \geq \mathcal{G}(\lambda_1^- + \sum_{k=1}^p \mu_{-1k}^-) - \mathcal{G}(\lambda_1^+ + \sum_{k=1}^p \mu_{1k}^+) + \frac{1}{2}(|\lambda_0| + \sum_{k=1}^p |\mu_{0k}|)$$

then the unique solution of (2.8) satisfies $u \geq 0$.

Proof: From the estimates of $G$ and $H$ given in (2.5) and the representation of the solution (2.4) we have

$$u(t) = \int_0^T G(t, s)[\sigma^+(s) - \sigma^-(s)]ds + G(t, 0)(\lambda_1^+ - \lambda_1^-) + H(t, 0)(\lambda_0^+ - \lambda_0^-)$$

$$+ \sum_{k=1}^p \left[ G(t, t_k)(\mu_{1k}^+ - \mu_{1k}^-) + H(t, t_k)(\mu_{0k}^+ - \mu_{0k}^-) \right]$$

$$\leq \mathcal{G} \int_0^T \sigma^+(s) ds + \lambda_1^+ + \sum_{k=1}^p \mu_{1k}^+ + \frac{1}{2}(|\lambda_0| + \sum_{k=1}^p |\mu_{0k}|)$$

$$- \mathcal{G} \int_0^T \sigma^-(s) ds + \lambda_1^- + \sum_{k=1}^p \mu_{1k}^- \leq 0, \quad t \in J,$$
and (i) holds. Analogously, we have

\[
\begin{align*}
\int_0^T G(t,s)\left[\sigma^+(s) - \sigma^-(s)\right]ds + G(t,0)(\lambda_1^+ - \lambda_1^-) + H(t,0)(\lambda_0^+ - \lambda_0^-)
& \quad + \sum_{k=1}^p \left[ G(t, t_k)(\mu_{1k}^+ - \mu_{1k}^-) + H(t, t_k)(\mu_{0k}^+ - \mu_{0k}^-) \right]
\geq G\left(\int_0^T \sigma^+(s)ds + \lambda_1^+ + \sum_{k=1}^p \mu_{1k}^+ \right) - \frac{1}{2}\left( |\lambda_0| + \sum_{k=1}^p |\mu_{0k}| \right) \\
& \quad - \overline{G}\left(\int_0^T \sigma^-(s)ds + \lambda_1^- + \sum_{k=1}^p \mu_{1k}^- \right) \geq 0, \quad t \in J,
\end{align*}
\]

and (ii) is valid. \(\square\)

We obtain as a corollary the following result that appears in [4].

**Corollary 2.2** If in problem (2.8) we have that \(\sigma \geq 0, \lambda_0 = 0, \lambda_1 \geq 0\) and \(\mu_{0k} = 0, \mu_{1k} \geq 0, k = 1, 2, \ldots, p\), then the unique solution of (2.8) satisfies \(u \leq 0\).

### 3 Monotone Iterative Technique

To develop the monotone method for problem (1.2) we introduce some hypotheses on \(f\) and \(I_k\).

(H1) There exists \(M > 0\) such that \(f\) satisfies the following one-side Lipschitz condition

\[
f(t, x) - f(t, y) \leq M^2(x - y)
\]
a.e. \(t \in J\) and \(\alpha(t) \leq y \leq x \leq \beta(t)\).

(H2) For each \(k \in \{1, 2, \ldots, p\}\), the impulsive perturbation \(I_k\) is continuous and nonincreasing.

Now, we introduce new definitions of lower and upper solutions for (1.2).

**Definition 3.1** We say that \(\alpha \in \Omega^{2,1}(J)\) is a lower solution for the problem (1.2) if there exist real constants \(\theta_0, \theta_1, e_{0k}, e_{1k}, k = 1, 2, \ldots, p\), such that

\[
\begin{align*}
\Delta \alpha'(t_k) &= I_k(\alpha(t_k)) + e_{1k}, \quad k = 1, 2, \ldots, p \\
\Delta \alpha(t_k) &= e_{0k} + \mu_{0k}, \quad k = 1, 2, \ldots, p \\
\alpha(0) - \alpha(T) &= \theta_0 + \lambda_0, \quad \alpha'(0) - \alpha'(T) = \theta_1 + \lambda_1,
\end{align*}
\]
with
\[\mathcal{G}\int_0^T (\alpha''(s) - f(s, \alpha(s))^+ ds - \mathcal{G}\int_0^T (\alpha''(s) - f(s, \alpha(s))^+ ds \leq \mathcal{G}(\gamma_1^- + \sum_{k=1}^p e_{1k}^-) - \mathcal{G}(\gamma_1^+ + \sum_{k=1}^p e_{1k}^+) - \frac{1}{2}(|\theta_0| + \sum_{k=1}^p |e_{0k}|).\]

Analogously,

**Definition 3.2** We say that \(\beta \in \Omega^{2,1}(J)\) is an upper solution for the problem (1.2) if there exist real constants \(\gamma_0, \gamma_1, \gamma_{0k}, \gamma_{1k}, k = 1, 2, \ldots, p\), such that
\[
\Delta \beta(t) = I_k(\beta(t)) + r_{1k}, \quad k = 1, 2, \ldots, p
\]
\[
\Delta \alpha(t) = r_{0k} + \mu_{0k}, \quad k = 1, 2, \ldots, p
\]
\[
\beta(0) - \beta(T) = \gamma_0 + \lambda_0, \quad \beta'(0) - \beta'(T) = \gamma_1 + \lambda_1,
\]

with
\[\mathcal{G}\int_0^T (f(s, \alpha(s)) - \beta''(s))^+ ds - \mathcal{G}\int_0^T (f(s, \beta(s)) - \beta''(s))^+ ds \leq -\mathcal{G}(\gamma_1^- + \sum_{k=1}^p r_{1k}^-) + \mathcal{G}(\gamma_1^+ + \sum_{k=1}^p r_{1k}^+) + \frac{1}{2}(|\theta_0| + \sum_{k=1}^p |e_{0k}|).\]

It is easy to see that these definitions generalize classical definitions that we expose in the first section.

Moreover, note that with Definition 3.1 it is possible, for example, that
\[\alpha''(t) < f(t, \alpha(t)),\]
in a set of measure positive, or
\[\Delta \alpha(t_k) < I_k(\alpha(t_k)),\]
for some \(k = 1, 2, \ldots, p\), oposite than in the classical definition.

**Theorem 3.1** Suppose that \((H1)\) and \((H2)\) hold and that there exist \(\alpha, \beta\) lower and upper solutions of the problem (1.2) in the sense of Definitions 3.1 and 3.2 respectively, and such that \(\alpha(t) \leq \beta(t)\) for each \(t \in J\). Then there exist two monotone sequences \(\{\alpha_n\}\) and \(\{\beta_n\}\) with \(\alpha_0 = \alpha, \beta_0 = \beta\) and
\[\alpha_{n-1} \leq \alpha_n \leq \beta_n \leq \beta_{n-1}, \quad n \geq 1,
\]
that converge uniformly to the minimal and maximal solutions of (1.2) in the sector \([\alpha, \beta] = \{z \in PC^1(J) : \alpha \leq z \leq \beta\}.

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Proof: For each \( \eta \in [\alpha, \beta] \), we define the problem.

\[
\begin{align*}
\frac{d^2 u}{dt^2} - M^2 u(t) &= f(t, \eta(t)) - M^2 \eta(t), & \text{a.e. } t \in J' \\
\Delta u(t_k) &= I_k(\eta(t_k)), & k = 1, 2, \ldots, p \\
\Delta u(t_k) &= \mu_k, & k = 1, 2, \ldots, p \\
u(0) - u(T) &= \lambda_0, & u'(0) - u'(T) = \lambda_1.
\end{align*}
\] (3.9)

By Corollary 2.1 we have that (3.9) has a unique solution \( u_\eta = u \). Therefore it is possible to define the following operator

\[ A : [\alpha, \beta] \subset PC^1(J) \rightarrow \Omega^{2,1}(J) \subset PC^1(J) \]

that if \( \eta \in L^1(J) \) then \( A\eta \equiv u_\eta \) with \( u_\eta \) the solution of the modified problem (3.9).

The operator \( A \) is continuous. We note that \( \xi \) is a solution of (1.1) if and only if \( \xi = A\xi \).

We claim that \( A \) satisfies the following two properties:

(i) \( \alpha \leq \eta_1 \leq \eta_2 \leq \beta \Rightarrow A\eta_1 \leq A\eta_2, \quad \eta_1, \eta_2 \in L^1(J) \),
(ii) \( \alpha \leq \eta \leq \beta \Rightarrow \alpha \leq A\eta \leq \beta, \quad \eta \in L^1(J) \).

To show (i) we define \( v = A\eta_1 - A\eta_2 \), thus for

\[
\frac{d^2 v}{dt^2} - M^2 v(t) = f(t, \eta_1(t)) - f(t, \eta_2(t)) - M^2 (\eta_1(t) - \eta_2(t)) \geq 0, \quad \text{a.e. } t \in J'
\]

and

\[
\begin{align*}
\Delta v(t_k) &= I_k(\eta_1(t_k)) - I_k(\eta_2(t_k)) \geq 0, & k = 1, 2, \ldots, p \\
\Delta v(t_k) &= 0, & k = 1, 2, \ldots, p \\
v(0) - v(T) &= 0, & v'(0) - v'(T) = 0,
\end{align*}
\]

thus, using Corollary 2.2 we have that \( v = A\eta_1 - A\eta_2 \leq 0 \) and (i) holds.

Now, to show (ii) we consider \( v = \alpha - A\eta \), and

\[
\begin{align*}
\frac{d^2 v}{dt^2} - M^2 v(t) &= \alpha''(t) - f(t, \alpha(t)) + f(t, \alpha(t)) - M^2 \alpha(t) \\
&= \alpha''(t) - f(t, \eta(t)) + M^2 \eta(t) \\
&\geq \alpha''(t) - f(t, \alpha(t)), \quad \text{a.e. } t \in J'
\end{align*}
\]

and

\[
\begin{align*}
\Delta v(t_k) &= I_k(\alpha(t_k)) - I_k(\eta(t_k)) + e_{1k} \geq e_{1k}, & k = 1, 2, \ldots, p \\
\Delta v(t_k) &= e_{0k}, & k = 1, 2, \ldots, p \\
v(0) - v(T) &= \theta_0, & v'(0) - v'(T) = \theta_1,
\end{align*}
\]
so that using expression (2.7), we know that \( v \leq w \) with \( w \) the unique solution of problem

\[
\begin{align*}
    \dddot{w}(t) - M^2 w(t) &= \alpha''(t) - f(t, \alpha(t)), \quad \text{a.e. } t \in J' \\
    \Delta w(t_k) &= e_{1k}, \quad k = 1, 2, \ldots, p \\
    \Delta w(t_k) &= e_{0k}, \quad k = 1, 2, \ldots, p \\
    w(0) - w(T) &= \theta_0, \quad w'(0) - w'(T) = \theta_1.
\end{align*}
\]

Now, using Lemma 2.1 we have that \( w \leq 0 \), and therefore \( \alpha \leq A\theta \) on \( J \).

Analogously we can show that \( A\theta \leq \beta \), and (ii) holds.

We define two sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) by:

\[
\begin{align*}
    \alpha_0 &= \alpha, \quad \beta_0 = \beta \quad &n = 0, \\
    \alpha_n &= A\alpha_{n-1}, \quad \beta_n = A\beta_{n-1}, \quad n \geq 1.
\end{align*}
\]

It is possible to give an explicit expression of terms of these sequences, for this we introduce

\[
\tilde{\alpha}_i(t) = f(t, \alpha_i(t)) - M^2 \alpha_i(t), \quad \text{a.e. } t \in J, \quad \text{for } i \in \mathbb{N},
\]

and

\[
\tilde{\beta}_i(t) = f(t, \beta_i(t)) - M^2 \beta_i(t), \quad \text{a.e. } t \in J, \quad \text{for } i \in \mathbb{N}.
\]

Then

\[
\begin{align*}
    \alpha_n(t) &= \int_0^T G(t, s)\tilde{\alpha}_{n-1}(s)ds + G(t, 0)\lambda_1 + H(t, 0)\lambda_0 \\
    &\quad + \sum_{k=1}^p [G(t, t_k)I_k(\alpha_{n-1}(t_k)) + H(t, t_k)\mu_{0k}]
\end{align*}
\]

and

\[
\begin{align*}
    \beta_n(t) &= \int_0^T G(t, s)\tilde{\beta}_{n-1}(s)ds + G(t, 0)\lambda_1 + H(t, 0)\lambda_0 \\
    &\quad + \sum_{k=1}^p [G(t, t_k)I_k(\beta_{n-1}(t_k)) + H(t, t_k)\mu_{0k}].
\end{align*}
\]

Now, the sets \( \{\alpha_n : n \in \mathbb{N}\} \), \( \{\beta_n : n \in \mathbb{N}\} \) are relatively compact sets in \( PC(J) \) since they are bounded and quasiequicontinuous [8, 16]. Then, there exist two functions \( \rho \) and \( \phi \) such that \( \{\alpha_n\} \to \rho \) and \( \{\beta_n\} \to \phi \) in \( PC(J) \) when \( n \to \infty \).

Now by the integral representation of the iterates \( \alpha_n \) and \( \beta_n \) we have that the limit functions belong to \( \Omega^{2,1}(J) \) and they are solutions of (1.2).

Finally to show that \( \rho \) and \( \phi \) are the minimal and maximal solutions of (1.2) in the sector \([\alpha, \beta]\) we consider \( u \in [\alpha, \beta] \) solution of (1.2), then using Lemma 2.1 it is easy to show that

\[
\begin{align*}
    \alpha_n \leq u \leq \beta_n &\Rightarrow \alpha_{n+1} \leq u \leq \beta_{n+1}, \quad n \geq 0.
\end{align*}
\]
Then passing to the limit when \( n \to \infty \) we obtain \( \rho \leq u \leq \phi \) on \( J \).

\[ \square \]

4 Discontinuous right side

In this section we study the existence of extremal solutions for our problem when \( f \) and \( I_k \) are not continuous functions. For this purpose, we consider the problem

\[
\begin{align*}
    u''(t) &= f(t, u(t), u(t)), & \text{a.e. } t \in J', \\
    \Delta u'(t_k) &= I_k(u(t_k), u(t_k)), & k = 1, 2, \ldots, p \\
    \Delta u(t_k) &= \mu_0, & k = 1, 2, \ldots, p \\
    u(0) - u(T) &= \lambda_0, & u'(0) - u'(T) = \lambda_1,
\end{align*}
\]

with \( \mu_0 \in \mathbb{R} \), for each \( k = 1, 2, \ldots, p \), \( I_k: \mathbb{R}^2 \to \mathbb{R} \) satisfies for \( k = 1, 2, \ldots, p \)

that \( I_k(x, y) \) is continuous in the first variable and nonincreasing in the second variable and finally \( f: J \times \mathbb{R}^2 \to \mathbb{R} \) is such that

\( f(\cdot, x, y(\cdot)) \) is measurable in \( J \) for each \( x \in \mathbb{R} \) and \( y \in PC^1(J) \).

\( f(t, \cdot, y) \) is continuous for a.e. \( t \in J \) and for each \( y \in \mathbb{R} \).

\( f(t, x, \cdot) \) is nonincreasing for a.e. \( t \in J \) and for each \( x \in \mathbb{R} \).

For every \( R > 0 \) there exists \( h_R \in L^1(J) \) such that \( |f(t, x, y)| \leq h_R(t) \) for a.e. \( t \in J \) and every \( x, y \in \mathbb{R} \) with \( |x| \leq R \) and \( |y| \leq R \).

**Definition 4.1** \( \alpha \in \Omega^{2,1}(J) \) is a lower solution of (4.10) if \( f(\cdot, \alpha(\cdot), \alpha(\cdot)) \in L^1(J) \) and there exist real constants \( \theta_0, \theta_1, \epsilon_0, \epsilon_1, k = 1, 2, \ldots, p, \) such that

\[
\begin{align*}
    \Delta \alpha'(t_k) &= I_k(\alpha(t_k), \alpha(t_k)) + \epsilon_{1k}, & k = 1, 2, \ldots, p \\
    \Delta \alpha(t_k) &= \epsilon_0 + \mu_0, & k = 1, 2, \ldots, p \\
    \alpha(0) - \alpha(T) &= \theta_0 + \lambda_0, & \alpha'(0) - \alpha'(T) = \theta_1 + \lambda_1,
\end{align*}
\]

with

\[
G \int_0^T (\alpha''(s) - f(s, \alpha(s), \alpha(s)))^+ ds - G \int_0^T (\alpha''(s) - f(s, \alpha(s), \alpha(s)))^- ds \leq
\]

\[
G(\theta_1^+ + \sum_{k=1}^p \epsilon_{1k}^+) - G(\theta_1^- + \sum_{k=1}^p \epsilon_{1k}^-) - \frac{1}{2}(\theta_0 + \sum_{k=1}^p \epsilon_{0k}^-).
\]

Analogously, we say that \( \beta \in \Omega^{2,1}(J) \) is an upper solution of (4.10) if \( f(\cdot, \beta(\cdot), \beta(\cdot)) \in L^1(J) \) and there exist real constants \( \gamma_0, \gamma_1, \rho_0, \rho_1, k = 1, 2, \ldots, p, \) such that

\[
\begin{align*}
    \Delta \beta'(t_k) &= I_k(\beta(t_k), \beta(t_k)) + \rho_{1k}, & k = 1, 2, \ldots, p \\
    \Delta \beta(t_k) &= \rho_0 + \mu_0, & k = 1, 2, \ldots, p \\
    \beta(0) - \beta(T) &= \gamma_0 + \lambda_0, & \beta'(0) - \beta'(T) = \gamma_1 + \lambda_1,
\end{align*}
\]
with
\[ G \int_0^T (f(s, \beta(s)), \beta''(s))^+ ds - G \int_0^T (f(s, \beta(s)), \beta''(s))^+ ds \leq -G(\gamma_1^- + \sum_{k=1}^{p} r_{1k}^-) + G(\gamma_1^+ + \sum_{k=1}^{p} r_{1k}^+) + \frac{1}{2}(|\gamma_0| + \sum_{k=1}^{p} |\gamma_k|). \]

We enunciate the following result adapted to \( PC^1(J) \) from Theorem 1.4.7 in [6].

**Lemma 4.1** Assume that \( \mathcal{B} : [\alpha, \beta] \rightarrow [\alpha, \beta] \) is a nondecreasing operator with \([\alpha, \beta] \subset PC^1(J)\) a not empty interval.

If there exists a constant \( K > 0 \), such that \( |\mathcal{B}y'(t)| \leq K \), for all \( y \in [\alpha, \beta] \), and for \( t \in J \),
then \( \mathcal{B} \) has a minimal fixed point \( x_* \) and a maximal fixed point \( x^* \) in \([\alpha, \beta]\).

Furthermore, we consider the following conditions.

(H3) There exists \( M > 0 \) such that \( f \) satisfies the following one-side Lipschitz condition
\[ f(t, x_1, y) - f(t, x_2, y) \leq M^2 (x_1 - x_2) \]
a.e. \( t \in J \), \( \alpha(t) \leq x_2 \leq x_1 \leq \beta(t) \) and \( y \in \mathbb{R} \).

(H4) For each \( k \in \{1, 2, \ldots, p\} \), the impulsive perturbation \( I_k \) is nonincreasing in the first variable.

**Theorem 4.1** Let \( \alpha, \beta \) lower and upper solutions of (4.10) such that \( \alpha \leq \beta \) in \( J \). Suppose that (H3) and (H4) hold. Then problem (4.10) has extremal solutions on the sector \([\alpha, \beta]\).

**Proof:** For \( y \in PC^1(J) \) consider the problem
\[
(P_y) \begin{cases}
  u''(t) = F_y(t, u(t)), & \text{a.e. } t \in J', \\
  \Delta u'(t_k) = I^y_k(u(t_k)), & k = 1, 2, \ldots, p \\
  \Delta u(t_k) = \mu_{yk}, & k = 1, 2, \ldots, p \\
  u(0) - u(T) = \lambda_0, & u'(0) - u'(T) = \lambda_1,
\end{cases}
\]
where \( F_y(t, x) = f(t, x, y(t)) \) and \( I^y_k(x) = I_k(x, y(t)) \), \( k = 1, 2, \ldots, p \).

Using the monotonicity of \( f \) and \( I_k \) it is easy to prove that \( \alpha \) and \( \beta \) are a lower and an upper solution respectively for \((P_y)\) for all \( y \in [\alpha, \beta] \) in the sense of Definitions 3.1 and 3.2.

Since (H3) and (H4) hold then \( F_y \) satisfies (H1) and functions \( I^y_k \) satisfy (H2) for all \( y \in [\alpha, \beta] \).
Thus, by Theorem 3.1 we have that for each \( y \in [\alpha, \beta] \) problem (\( P_y \)) has extremal solutions in \([\alpha, \beta]\).

Define \( \mathcal{B} : [\alpha, \beta] \to [\alpha, \beta] \) by \( \mathcal{B}y = u \), where \( u \) is the maximal solution of (\( P_y \)) for each \( y \in [\alpha, \beta] \).

\( \mathcal{B} \) is a nondecreasing map since if \( y_1, y_2 \in [\alpha, \beta] \), \( y_1 \leq y_2 \), and \( u_1 = \mathcal{B}y_1 \), \( u_2 = \mathcal{B}y_2 \), for a.e. \( t \in J' \) we have

\[
u_i'(t) = F_{y_i}(t, u_i(t)) = f(t, u_i(t), y_i(t)) \leq f(t, u_1(t), y_1(t)) = \nu_1'(t),
\]

and

\[
\Delta u_1(t_k) = I_k^y(u_1(t_k)) = I_k(u_1(t_k), y_1(t_k)) \\
\geq I_k(u_1(t_k), y_2(t_k)) = I_k^y(u_1(t_k)), \quad k = 1, 2, \ldots, p.
\]

Thus, \( u_1 \) is a lower solution for the problem (\( P_{y_2} \)) and Theorem 3.1 implies the existence of extremal solutions of (\( P_{y_2} \)) in \([u_1, \beta] \). But \( u_2 \) is the maximal solution of (\( P_{y_2} \)) in \([\alpha, \beta] \), thus \( \mathcal{B}y_1 = u_1 \leq u_2 = \mathcal{B}y_2 \).

On the other hand, for each \( t \in J \)

\[
(By)'(t) = \left| \int_0^T H(t, s)[F_y(s, [By](s)) - M^2|By|(s)]ds + H(t, 0)\lambda \right|
\]

\[
+M^2G(t, 0)\lambda_0 + \sum_{k=1}^p \left( H(t, t_k)I_k^y([By](t_k)) + M^2G(t, t_k)\mu_{0k} \right),
\]

but by hypothesis there exists \( h \in L^1(J) \) such that

\[
|F_y(t, x)| \leq h(t)
\]

for all \( y \in [\alpha, \beta] \) and with \( x \in [\min_{t \in J} \alpha(t), \max_{t \in J} \beta(t)] \subset [-R_1, R_1] \), for some \( R_1 > 0 \). And

\[
|I_k^y(x)| \leq R_2
\]

since \( |I_k^y(\beta(t_k)), I_k^y(\alpha(t_k))| \subset [-R_2, R_2], k = 1, 2, \ldots, p \), for some \( R_2 > 0 \). Thus

\[
|By'(t)| \leq \frac{1}{2} \int_0^T h(s)ds + M^2TR_1 + \frac{1}{2} \lambda
\]

\[
+M^2G\lambda_0 + \sum_{k=1}^p \left( \frac{R_2}{2} + M^2G\mu_{0k} \right) \leq \mathcal{K},
\]

and by Lemma 4.1 we obtain that \( \mathcal{B} \) has a maximal fixed point on \([\alpha, \beta]\) which, by definition of \( \mathcal{B} \), is the maximal solution of (4.10) in \([\alpha, \beta]\).

Analogously, to show the existence of the minimal solution of (4.10) in \([\alpha, \beta]\) we consider the operator \( \bar{\mathcal{B}} \) that for each \( y \in [\alpha, \beta] \) gives the minimal solution of the problem (\( P_y \)) in \([\alpha, \beta] \). \( \square \)
Dynamics of continuous, discrete and impulsive systems, 7 (2000) 145-158.15

Acknowledgements

Research partially supported by Ministerio de Educació n y Cultura, DGESIC, project PB97-0552, by Xunta de Galicia, project XUGA20701B98 and by INTAS, project 96-0915.

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References


Dynamics of continuous, discrete and impulsive systems, 7 (2000) 145-158


